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Non-commutative Orlicz spaces associated to a modular on τ-measurable operators

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ABSTRACT

In this paper, we consider non-commutative Orlicz spaces as modular spaces and show that they are complete with respect to their modular. We prove some convergence theorems for τ-measurable operators and deal with uniform convexity of non-commutative Orlicz spaces.

1. Introduction

The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [1] and were intensively developed by his mathematical school: Amemiya, Koshi, Shimogaki, Yamamuro [2,3] and others. Further and the most complete development of these theories are due to Orlicz, Mazur, Musielak, Luxemburg, Turpin [4] and their collaborators. At present the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [5] and interpolation theory [6], which in their turn have broad applications [4,7]. The importance for applications consists in the richness of the structure of modular function spaces, that – besides being Banach spaces (or F-spaces in a more general setting) – are equipped with modular equivalent of norm or metric notions.

This article is devoted to a study of some properties of non-commutative Orlicz spaces. Non-commutative Orlicz spaces can be defined either in an algebraic way [8] or via Banach function spaces [9,10]. Al-Rashed and Zegarliński [11] established the theory of non-commutative Orlicz spaces associated to a non-commutative Orlicz functional. Their non-commutative Orlicz functional is related to those introduced by [12] where the author used a specific Young function $\varphi(x) = \cosh(x) - 1$, which has a particular importance in quantum information geometry. Recently, they investigated a theory associated with a faithful normal state on a semi-finite von Neumann algebra [13]. Some further results to non-commutative Orlicz spaces are due to Muratov [14]. We consider another approach based on the concept of modular function spaces. Using the generalized singular value function of a τ-measurable operator, we define a modular on the collection of all τ-measurable operators. This modular function defines a corresponding modular space, which is called the non-commutative Orlicz space.

The organization of the paper is as follows. In the second section we provide some necessary preliminaries related to the theory of τ-measurable operators affiliated with a von Neumann algebra and the classical theory of modular spaces. In Section 3 we introduce a definition of non-commutative Orlicz spaces associated to a modular on τ-measurable operators

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and give several equivalent norms on such spaces. We show that the non-commutative Orlicz space is complete with respect to the modular. Finally, in Section 4 we prove that the non-commutative Orlicz space $L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ is uniformly convex if the Orlicz function φ is uniformly convex and satisfies the Δ_2-condition.

2. Preliminaries

In this section, we collect some basic facts and introduce some notation related to τ-measurable operators and modular function spaces. We denote by $\mathfrak{M}$ a semi-finite von Neumann algebra on a Hilbert space $\mathfrak{H}$, with a fixed faithful and normal semi-finite trace τ. For standard facts concerning von Neumann algebras, we refer the reader to [15,16]. The identity in $\mathfrak{M}$ is denoted by $\mathbf{1}$ and we denote by $\mathcal{P}(\mathfrak{M})$ the complete lattice of all self-adjoint projections in $\mathfrak{M}$. A linear operator $x : \mathcal{D}(x) \to \mathfrak{H}$ with domain $\mathcal{D}(x) \subseteq \mathfrak{H}$ is called affiliated with $\mathfrak{M}$, if $ux = xu$ for all unitaries u in the commutant $\mathfrak{M}'$ of $\mathfrak{M}$. This is denoted by $x\eta\mathfrak{M}$. Note that the equality $ux = xu$ involves the equality of the domains of the operators ux and xu, that is, $\mathcal{D}(x) = u^{-1}(\mathcal{D}(x))$. If x is in the algebra $\mathcal{B}(\mathfrak{H})$ of all bounded linear operators on the Hilbert space $\mathfrak{H}$, then x is affiliated with $\mathfrak{M}$ if and only if $x \in \mathfrak{M}$. If x is a self-adjoint operator in $\mathcal{B}(\mathfrak{H})$ affiliated with $\mathfrak{M}$, then the spectral projection $e^x(B)$ is an element of $\mathfrak{M}$ for any Borel set $B \subseteq \mathbb{R}$.

A closed and densely defined operator x, affiliated with $\mathfrak{M}$, is called τ-measurable if and only if there exists a number $\lambda \geq 0$ such that

$$\tau\left(e^{|x|}(\lambda, \infty)\right) < \infty.$$

The collection of all τ-measurable operators is denoted by $\widetilde{\mathfrak{M}}$. With the sum and product defined as the respective closure of the algebraic sum and product, it is well known that $\widetilde{\mathfrak{M}}$ is a $*$-algebra [17]. Given $0 < \varepsilon, \delta \in \mathbb{R}$, we define $\mathcal{V}(\varepsilon, \delta)$ to be the set of all $x \in \widetilde{\mathfrak{M}}$ for which there exists $p \in \mathcal{P}(\mathfrak{M})$ such that $\|xp\|_{\mathcal{B}(\mathfrak{H})} \leq \varepsilon$ and $\tau(\mathbf{1} - p) \leq \delta$. An alternative description of this set is given by

$$\mathcal{V}(\varepsilon, \delta) = \left\{x \in \widetilde{\mathfrak{M}} \quad : \tau\left(e^{|x|}(\varepsilon, \infty)\right) < \delta\right\}.$$

The collection $\{\mathcal{V}(\varepsilon, \delta)\}_{\varepsilon,\delta>0}$ is a neighborhood base at 0 for a vector space topology τ_m on $\widetilde{\mathfrak{M}}$. For $x \in \widetilde{\mathfrak{M}}$, the generalized singular value function $\mu(x; \cdot) = \mu(|x|; \cdot)$ is defined by

$$\mu(x; t) = \inf\left\{\lambda \geq 0 : \quad \tau\left(e^{|x|}(\lambda, \infty)\right) \leq t\right\}, \quad t \geq 0.$$

It follows directly that the generalized singular value function $\mu(x)$ is a decreasing right-continuous function on the positive half-line $[0, \infty)$. Moreover,

$$\mu(uxv) \leq \|u\|\|v\|\mu(x)$$

for all $u, v \in \mathfrak{M}$, and $x \in \widetilde{\mathfrak{M}}$ as well as

$$\mu(f(x)) = f(\mu(x))$$

whenever $0 \leq x \in \widetilde{\mathfrak{M}}$ and f is an increasing continuous function on $[0, \infty)$, which is satisfying $f(0) = 0$. The space $\widetilde{\mathfrak{M}}$ is a partially ordered vector space under the ordering $x \geq 0$ defined by $\langle x\xi, \xi\rangle \geq 0, \xi \in \mathcal{D}(x)$. If $0 \leq x_\alpha \uparrow x$ holds in $\widetilde{\mathfrak{M}}$, then $\sup \mu(x_\alpha; t)\uparrow_\alpha \mu(x; t)$ for each $t \geq 0$. The trace τ is extended to the positive cone of $\widetilde{\mathfrak{M}}$ as a non-negative extended real-valued functional which is positively homogeneous, additive, unitarily invariant and normal. Furthermore,

$$\tau(x^*x) = \tau(xx^*)$$

for all $x \in \widetilde{\mathfrak{M}}$ and

$$\tau(f(x)) = \int_0^\infty f(\mu(x; t))dt \tag{2.1}$$

whenever $0 \leq x \in \widetilde{\mathfrak{M}}$ and f is a non-negative Borel function, which is bounded on a neighborhood of 0 and satisfies $f(0) = 0$. The generalized singular value functions are the analog (and, actually, generalization) of the decreasing rearrangements of functions in the classical setting. In the following proposition, we list some properties of the rearrangement mapping $\mu(\cdot; t)$.

Proposition 2.1. *Let x, y and z be τ-measurable operators.*

(i) *the map $t \in (0, \infty) \mapsto \mu(x; t)$ is non-increasing and continuous from the right. Moreover,*

$$\lim_{t\downarrow 0} \mu(x; t) = \|x\| \in [0, \infty].$$

(ii) $\mu(x; t) = \mu(|x|; t) = \mu(x^*; t)$.
(iii) $\mu(x; t) \leq \mu(y; t),\ t > 0$, *if* $0 \leq x \leq y$.
(iv) $\mu(x + y; t + s) \leq \mu(x; t) + \mu(y; s),\ t, s > 0$.
(v) $\mu(zxy; t) \leq \|z\|\|y\|\mu(x; t),\ t > 0$.
(vi) $\mu(xy; t + s) \leq \mu(x; t)\mu(y; s)\, t, s > 0$.