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Symmetries of the Fisher–Kolmogorov–Petrovskii–Piskunov equation with a nonlocal nonlinearity in a semiclassical approximation

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ABSTRACT

The classical group analysis approach used to study the symmetries of integro-differential equations in a semiclassical approximation is considered for a class of nearly linear integro-differential equations. In a semiclassical approximation, an original integro-differential equation leads to a finite consistent system of differential equations whose symmetries can be calculated by performing standard group analysis.

The approach is illustrated by the calculation of the Lie symmetries in explicit form for a special case of the one-dimensional nonlocal Fisher–Kolmogorov–Petrovskii–Piskunov population equation.

1. Introduction

The mathematical models used for studying nonlocal interactions in physical, chemical, and biological systems are based on nonlinear integro-differential equations (IDEs).

The important IDEs widely used in applications are kinetic equations. A detailed review of kinetic phenomena and their modeling in plasma physics, in rarefied gas dynamics, and the other physical systems can be found elsewhere [1,2].

The theory of Bose–Einstein condensates substantially employs the Gross–Pitaevskii equation (GPE) [3]. The nonlocal GPE describes the evolution of coherent quantum ensembles of dipolar quantum gases with long-range dipole–dipole interaction which gives rise to novel properties of quantum matter (see, e.g., [4], and references therein).

The Fokker–Planck equation with a nonlocal nonlinearity was used in a stochastic theory involving feedback and a nonlinear family of Markov diffusion processes [5].

The classical Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) population equation [6,7] has been used in mathematical biology to explain the space–time evolution of microbiological population densities (bacteria or cells) due to the diffusion mechanism. To take into consideration long-range interactions of individuals typical of the colonial organization of microbial populations [8], nonlocal generalizations of the FKPP equation are used [9]. Nonlocal models are aimed, in particular, at describing the pattern formation in bacterial colonies [9]. This contributes to the study of micro-morphogenesis, which is of particular interest in the fundamental problems of modern microbiology [8].

The symmetry groups of IDEs are calculated by direct or indirect methods [10]. Algorithms of indirect calculation are based on replacing the input nonlocal IDE with a system of partial differential equations (PDEs). The system is then analyzed by the standard methods of the classical Lie group analysis of PDEs [11–14]. Nonlocal equations can be reduced to PDEs by

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using the method of moments or a covering method, amongst others (see, e.g., [10]). The first method was used to calculate the Lie point symmetry group for the Vlasov–Maxwell equations in plasma theory [15] and for the Benney, Vlasov-type, and Boltzmann-type kinetic equations [16]. The covering method was developed [17] and applied to a coagulation kinetic equation.

Direct methods were developed and also applied to the Boltzmann equation, equations of motion of viscoelastic media, the Benney, and the Vlasov–Maxwell equations (see [1,10,18,19] and references therein).

Advances in group analysis were achieved in the theory of one-parameter and multi-parameter approximate transformation groups initiated by Baikov et al. [20]. Similar ideas were suggested by Fushchych et al. [21]. Approximate symmetries involve a small parameter and are calculated for PDEs with or without a small parameter. The approximate symmetries were found, for example, for the Boussinesq equation [22], for nonlinear wave equations [23], and for other types of equation [20].

The area of nonlocal methods has attracted research because of new prospects for the development of symmetry analysis. Back in the 1990s, Fushchych and Shtelen [24] considered nonlocal transformations generated by linear differential operators. Fushchych et al. [25] used nonlocal ansätze to reduce nonlinear PDEs to equations with a lower number of independent variables. The nonlocal ansätze were shown to relate to conditional (nonclassical) symmetries of PDEs [26–29]. Akhatov et al. studied contact and quasilocal symmetries for nonlinear diffusion equations [30].

These ideas have been developed by Bluman et al. [31], Bluman and Cheviakov [32], Popovych et al. [33], Kunzinger and Popovych [34], Boyko [35], Zhdanov [36], and others.

Note that, in all the above references, the approximate symmetries and solutions of the equations under consideration were regularly dependent on a small parameter.

However, the same operator is known to possess different properties in different classes of functions (e.g., a differentiation operator is bounded on $\mathcal{C}_2$ and unbounded on L_2). As a consequence, the symmetries themselves and the way of their calculation depend on the class of functions for which the equation operator is defined. This leads to the idea of defining the equation operator in a class of functions singularly depending on a small asymptotic parameter and of finding the relevant approximate solutions, symmetries, conservation laws, and symmetry operators.

Such classes of functions are used to construct solutions of PDEs in a semiclassical approximation in the context of the Maslov canonical operator method [37], of the complex germ method [38,39], or of the generalized adiabatic method [40].

The main idea of this paper is to consider a technique which admits classical group analysis methods to be applied to study the symmetries of IDEs with the use of a semiclassical approximation. We consider here a special class of equations: nearly linear IDEs. In a semiclassical approximation, an original IDE leads to a finite consistent system of differential equations whose symmetries can be investigated by standard group analysis [11–14].

In the next section, the construction of a consistent finite system is considered in the framework of a semiclassical approximation for a nearly linear one-dimensional IDE of general form.

The scheme for calculating the symmetries of the consistent system is given in Section 2, where the integral constraints that arise in this approach are also discussed.

In Sections 3 and 4, the general ideas are illustrated by a simple but nontrivial example of a one-dimensional nonlocal FKPP equation of particular type. The Lie symmetries and the corresponding similarity solution are found in explicit form.

2. The consistent system and a semiclassical approximation

Consider an rth-order evolution IDE with a small asymptotic parameter ε as a factor of the partial derivatives, i.e.,

$$\hat{L}[u](t,x,\varepsilon)=0, \tag{2.1}$$

$$\hat{L}[u](t,x,\varepsilon)=-\varepsilon u_t(t,x)+\hat{F}[u,I](t,x,\varepsilon), \tag{2.2}$$

$$\hat{F}[u,I](t,x,\varepsilon)=F(t,x,u(t,x),\varepsilon u_x(t,x),\ldots,\varepsilon^r u_{xx\cdots x}(t,x);\hat{I}[u](t,x),\varepsilon), \tag{2.3}$$

where the smooth real scalar function $u(t,x)$ depends on time t and belongs to the Schwarz space $\mathcal{S}$ in the space variable x. Here $\hat{I}[u](t,x)=(\hat{I}_1[u](t,x),\ldots,\hat{I}_l[u](t,x))$,

$$\hat{I}_k[u](t,x)=\int_{-\infty}^{\infty}b_k(t,x,y)u(t,y)dy,\quad k=\overline{1,l}; \tag{2.4}$$

$b_k(t,x,y)$ is a smooth function of t,x,y growing with x,y no faster than a polynomial; $u_x=\partial u(t,x)/\partial x$; and $u_{xx}=\partial^2u(t,x)/\partial x^2,\ldots$

Denote by $t,x,u,u_1,u_2,\ldots,u_r;I_1,\ldots,I_l$ a set of real independent variables. A collection of variables $z_r=(x,u,u_1,u_2,\ldots,u_r)$ can be assigned to as a point of the jet space $J^{(r)}$ (see, e.g., [13]). The right-hand side of (2.3) is determined by a function

$$F(t,x,u,u_1,u_2,\ldots,u_r;I_1,\ldots,I_l)=F(t,z_r,I,\varepsilon), \tag{2.5}$$