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Global strong solutions of two-dimensional Navier–Stokes equations with nonlinear slip boundary conditions[☆]

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ABSTRACT

The two-dimensional time-dependent Navier–Stokes equations with nonlinear slip boundary conditions are investigated in this paper. Since the nonlinear slip boundary conditions of this type include the subdifferential property, the weak variational formulation is the variational inequality. The existence, uniqueness and regularity of global weak solutions are shown using the regularized method. Moreover, the continuous dependence property of the weak solution for given initial data and the behavior of the global weak solution as $t \longrightarrow +\infty$ are established.

1. Introduction

It is well known that the mathematical model of viscous incompressible fluid is given by the Navier–Stokes equations, which can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } Q_T, \\ \operatorname{div} u = 0 & \text{in } Q_T \end{cases} \tag{1}$$

where $Q_T = (0, T) \times \Omega$, $0 < T \leq +\infty$, $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ composed of two components Γ and S satisfying $\Gamma \cap S = \emptyset$, $|\Gamma| \neq 0$, $|S| \neq 0$ and $\overline{\Gamma \cup S} = \partial\Omega$. $u(t, x)$ and $f(t, x)$ are vector functions and denote the flow velocity and the external force, respectively. $p(t, x)$ is a scalar function and denotes the pressure. The viscous coefficient $\mu > 0$ is a positive constant. The solenoidal condition means that the fluid is incompressible.

To deal with (1), the proper boundary conditions must be attached. We know that different boundary conditions describe different physical phenomena. Fujita in [1] investigated some hydrodynamics problems under some nonlinear boundary conditions, such as leak and slip boundaries including the subdifferential property. Boundary conditions of this type appear in the modeling of blood flow in a vein of an arterial sclerosis patient and in the modeling of avalanches of water and rocks.

In this paper, we will consider the following initial–boundary value conditions:

$$\begin{cases} u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma \times (0, T], \\ u_n = 0,\ -\sigma_\tau(u) \in g\partial|u_\tau| & \text{on } S \times (0, T], \end{cases} \tag{2}$$

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where $u(0)$ denotes the initial value of $u(t)$ at $t = 0$, and g is a scalar function; $u_n = u \cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, where n stands for the unit vector of the external normal to S; $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of p, is the tangential component of the stress vector σ which is defined by $\sigma_i = \sigma_i(u, p) = (\mu e_{ij}(u) - p\delta_{ij})n_j$, where $e_{ij}(u) = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}$, $i, j = 1, 2$. The set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a, whose definition will be given in Section 2.

There are some theoretical results concerning viscous incompressible flow with nonlinear subdifferential boundary conditions. Fujita in [2] showed the existence and uniqueness of weak solutions to the Stokes problem with slip or leak boundary conditions. Subsequently, Saito in [3] showed the regularity of these weak solutions by Yosida's regularized method and the finite difference quotients method. Fujita obtained in [4,5], in terms of nonlinear semigroup theory, the well-posedness of the initial–boundary value Stokes problem with leak boundary conditions. Other results concerning the Stokes problems can be found in [6,7]. For three-dimensional steady Navier–Stokes equations, Chebotarev in [8] obtained the existence of weak solutions via a limited tangential component of velocity. For three-dimensional time-dependent nonlinear Navier–Stokes equations, Konovalova in [9] proved the weak solvability under the assumption that

$$|((u \cdot \nabla)u, v)| \leq c\|u\|^{1+\theta}\|\nabla u\|^{1-\theta}\|v\|,$$

where $c > 0$ is a constant and $\theta \in (0, 1)$. Furthermore, the solution is regular if

$$|((u \cdot \nabla)u, v)| \leq c\|u\|^{1+\theta}\|\nabla u\|^{1-\theta}\|\nabla v\|^{\gamma}\|v\|^{1-\gamma},$$

where $\theta, \gamma \in (0, 1/2)$, and $\|\cdot\|$ denotes the norm in $L^2(\Omega)$. Other results concerning Navier–Stokes equations with nonlinear subdifferential boundary conditions can be found in [10,11]. We remark that the steady homogeneous and inhomogeneous Stokes system with linear slip boundary conditions without a subdifferential property have recently been studied by Veiga in [12–14].

In this paper, we will deal with the two-dimensional Navier–Stokes equations (1) with nonlinear slip boundary conditions (2). Since nonlinear slip boundary conditions of this type include the subdifferential property, the weak variational formulation is the variational inequality problem. It is well known that the regularized method plays a key role in theoretical analysis and numerical analysis of the variational inequality problem because it reduces the inequality to the equation. Then many tools can be used to deal with the variational equation. Here, we use the regularized method to deal with the variational inequality problem. First, we obtain the regularized problem whose weak formulation is the variational equation. Subsequently, we show the existence and regularity of global weak solutions to the regularized problem by the Faedo–Galerkin method, and obtain the existence and regularity of weak solutions to the variational inequality problem as $\varepsilon \longrightarrow 0$. Finally, we establish the continuous dependence property of the weak solution for given initial data and the behavior of the global weak solution as $t \longrightarrow +\infty$. We will show that the global weak solution converges to the weak solution of the corresponding steady Navier–Stokes equations as $t \longrightarrow +\infty$. Thus, these results derived in this paper are similar to the well-posedness properties of the two-dimensional Navier–Stokes equations with complete homogeneous Dirichlet boundary conditions [15].

This paper is organized as follows. In the next section, we will define some spaces used usually and describe the definitions of the weak solution and the strong solution via a variational inequality. Moreover, the associated regularized problem is given. In Section 3, we will study the steady Stokes problem with nonlinear subdifferential boundary conditions and introduce the Stokes operator A. The existence, uniqueness and regularity of weak solutions to the variational inequality are shown by the regularized method in Section 4. The continuous dependence property of the weak solution for given initial data and the behavior of the global weak solution as $t \longrightarrow +\infty$ are established in the last two sections.

2. Navier–Stokes equations with nonlinear slip boundary conditions

First, we give the definition of the subdifferential property (e.g. [16]). Let $\psi : \mathbb{R}^2 \to \overline{\mathbb{R}} = (-\infty, +\infty]$ be a given function possessing the properties of convexity and weak semi-continuity from below (ψ is not identical to $+\infty$). The subdifferential set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a:

$$\partial\psi(a) = \{b \in \mathbb{R}^2 \mid \psi(h) - \psi(a) \geq b \cdot (h - a), \forall h \in \mathbb{R}^2\}.$$

We introduce some spaces which are usually used in this paper. Define

$$V = \{u \in H^1(\Omega)^2, u|_\Gamma = 0, u \cdot n|_S = 0\}, \qquad V_\sigma = \{u \in V, \operatorname{div} u = 0\},$$
$$H = \{u \in L^2(\Omega)^2, \operatorname{div} u = 0, u \cdot n|_{\partial\Omega} = 0\}, \qquad L_0^2(\Omega) = \left\{q \in L^2(\Omega), \int_\Omega q dx = 0\right\}.$$

Let $\|\cdot\|_k$ be the norm in Hilbert space $H^k(\Omega)^2$. Let $(\cdot, \cdot)$ and $\|\cdot\|$ be the inner product and the norm in $L^2(\Omega)^2$. Then we can equip V with the norm $\|v\|_V = \|\nabla v\|$ for all $v \in V$ because $\|\nabla v\|$ is equivalent to $\|v\|_1$ in view of the Poincaré inequality. Let V' be the dual space of V and $\langle\cdot, \cdot\rangle$ be the dual pairing in $V \times V'$.

If X is a Banach space, $L^p(0, T; X)$, $1 \leq p < +\infty$, is the linear space of measurable functions from the interval $[0, T]$ into X such that $\int_0^T \|u(t)\|_X^p dt < \infty$. If $p = +\infty$, we require that $\sup_{t\in[0,T]} \|u(t)\|_X < \infty$.