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The winding number of $Pf+1$ for polynomials P and meromorphic extendibility of f

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ABSTRACT

Let Δ be the open unit disc in $\mathbb{C}$. The paper deals with the following conjecture: if f is a continuous function on $b\Delta$ such that the change of argument of $Pf+1$ around $b\Delta$ is nonnegative for every polynomial P such that $Pf+1$ has no zero on $b\Delta$ then f extends holomorphically through Δ. We prove a related result on meromorphic extendibility for smooth functions with finitely many zeros of finite order, which, in particular, implies that the conjecture holds for real analytic functions.

1. Introduction

Let Δ be the open unit disc in $\mathbb{C}$. Given a continuous function φ on $b\Delta$ with no zero on $b\Delta$ we denote by $W(\varphi)$ the winding number of φ around 0, so $2\pi W(\varphi)$ is the change of argument of $\varphi(z)$ as z runs around $b\Delta$ in the positive direction. If a function g is holomorphic on Δ then we denote by $Z(g)$ the number of zeros of g counting multiplicity. We denote by $A(\Delta)$ the disc algebra, that is the algebra of all continuous functions on $\overline{\Delta}$ which are holomorphic on Δ. It is known that one can characterize holomorphic extendibility in terms of the argument principle:

Theorem 1.1 ([1–3]). *A continuous function f on $b\Delta$ extends holomorphically through Δ if and only if $W(f+Q)\geq 0$ for every polynomial Q such that $f+Q\neq 0$ on $b\Delta$.*

Note that the "only if" part is a consequence of the argument principle.

One can view $f+Q$ above as $Pf+Q$ with $P\equiv 1$. We believe that an analogous theorem holds for $Q\equiv 1$:

Conjecture 1.2. *Let f be a continuous function on $b\Delta$ such that*

$$W(Pf+1)\geq 0 \tag{1.1}$$

whenever P is a polynomial such that $Pf+1\neq 0$ on $b\Delta$. Then f extends holomorphically through Δ.

The present note is the result of an unsuccessful attempt to prove this conjecture. In the paper we prove the conjecture for sufficiently smooth functions with finitely many zeros of finite order. In particular, the conjecture holds for real analytic functions.

2. Functions with no zeros

Suppose that the function f has no zero. In this case (1.1) implies that $W(f)\geq 0$. Indeed, there is an $\varepsilon>0$ such that $W(f)=W(f+\eta)$ for all $\eta,\ |\eta|<\varepsilon$. Choosing for P a constant $c,\ |c|>1/\varepsilon$, (1.1) implies that $W(f)\geq 0$. Note that (1.1) implies that

$$W(f)+W\left(P+\frac{1}{f}\right)=W(Pf+1)\geq 0$$

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so

$$W(P+1/f) \geq -W(f)$$

for every polynomial P such that $P+1/f \neq 0$ on $b\Delta$. If $W(f)=0$ then Theorem 1.1 implies that $1/f$ extends holomorphically through Δ and since $W(f)=0$ the argument principle shows that this holomorphic extension has no zero on Δ which gives:

Proposition 2.1. *Let f be a continuous function on $b\Delta$ which has no zero and which satisfies $W(f)=0$. If $W(Pf+1) \geq 0$ whenever P is a polynomial such that $Pf+1 \neq 0$ on $b\Delta$ then f extends holomorphically through Δ.*

Now, let $W(f)=N>0$. Then $W(Pf+1)=W(f)+W(P+1/f) \geq 0$, so $W(1/f+P) \geq -N$ for every polynomial P such that $1/f+P \neq 0$ on $b\Delta$. A recent theorem of Raghupathi and Yattselev [4, Theorem 2] applies to show that if f is α-Hölder continuous with $\alpha > 1/2$ then $1/f$ has a meromorphic extension through Δ which has at most N poles, counting multiplicity. So in this case there are a function H in the disc algebra and a polynomial R of degree not exceeding N, with all zeros contained in Δ, such that

$$\frac{1}{f(z)} = \frac{H(z)}{R(z)} \quad (z \in b\Delta).$$

Since $W(1/f)=-N$ and since $\deg R \leq N$ it follows by the argument principle that R has exactly N zeros in Δ, counting multiplicity, and H has no zero on $\overline{\Delta}$. It follows that $f=R/H$ extends holomorphically through Δ. This proves:

Proposition 2.2. *Let f be an α-Hölder continuous function on $b\Delta$ with $\alpha > 1/2$ which has no zero on $b\Delta$. If $W(Pf+1) \geq 0$ whenever P is a polynomial such that $Pf+1 \neq 0$ on $b\Delta$ then f extends holomorphically through Δ.*

3. Functions with finitely many zeros

The reasoning in Section 2 is no longer possible if f has zeros on $b\Delta$. Suppose that f has the form

$$f(z) = (z-b_1)^{m_1}(z-b_2)^{m_2}\cdots(z-b_n)^{m_n}g(z) \quad (z \in b\Delta),$$

where $b_i \in b\Delta$, $1 \leq i \leq n$, $b_i \neq b_j$ $(i \neq j)$ and where g is a continuous function with no zeros. (In particular, this holds when f is real analytic). Assume that $W(g)=N$. Then (1.1) implies that

$$W(1/g + (z-b_1)^{m_1}\cdots(z-b_n)^{m_n}P) \geq -N.$$

If $m_1=\cdots=m_n=0$ as in the preceding section then, if $N \geq 0$, [4, Theorem 2] implies that $1/g$ has a meromorphic extension through Δ with at most N poles. So the relevant question now is whether the same is true in general:

Question 3.1. Let $b_1, b_2, \ldots, b_n \in b\Delta$, $b_i \neq b_j$ if $i \neq j$, and let $m_1, \ldots, m_n \in \mathbb{N}$. Let $\mathcal{P}$ be the family of all polynomials Q of the form

$$Q(z) = (z-b_1)^{m_1}\cdots(z-b_n)^{m_n}p(z),$$

where p is a polynomial, and let $J \in \mathbb{N}\cup\{0\}$. Suppose that f is a continuous function on $b\Delta$ such that

$$f(b_j) \neq 0 \quad (1 \leq j \leq n) \tag{3.1}$$

and such that $W(f+Q) \geq -J$ for each $Q \in \mathcal{P}$ such that $f+Q \neq 0$ on $b\Delta$. Must f extend meromorphically through Δ with the extension having at most J poles, counting multiplicity?

Note that one has to assume (3.1) since otherwise $W(f+Q)$ is undefined for every $Q \in \mathcal{P}$. If $m_1=\cdots=m_n=0$ and $J=0$ then the positive answer is provided by Theorem 1.1. If $m_1=\cdots=m_n=0$ and $J \geq 1$ and if f is α-Hölder continuous with $\alpha > 1/2$ then the answer is positive by [4, Theorem 2] which was proved by using the theorem on rigid interpolaton:

Theorem 3.2 ([4, Theorem 5])**.** *Suppose that g is a holomorphic function on Δ and let $N \in \mathbb{N}$. Suppose that for every nonnegative integer n and for every polynomial p of degree not exceeding n we have*

$$Z(z^n g + p) \leq N + n. \tag{3.2}$$

Then g is a quotient of polynomials of degree not exceeding N.

Note that if g is a quotient of polynomials of degree not exceeding N then (3.2) holds for every polynomial p of degree not exceeding n. In the present paper we use Theorem 3.2 as an essential tool.