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Strassen-type law of the iterated logarithm for self-normalized increments of sums

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ABSTRACT

Let $\{X, X_n, n \ge 1\}$ be a sequence of independent identically distributed random variables with $EX = 0$ and assume that $EX^2 I(|X| \le x)$ is slowly varying as $x \to \infty$, i.e., X is in the domain of attraction of the normal law. In this paper it is shown that a Strassen-type functional law of the iterated logarithm holds for self-normalized increments of sums of such random variables.

1. Introduction and main result

Let $\{X, X_n,\ n \ge 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables. Put $S_0 = 0$ and

$$S_n = \sum_{i=1}^n X_i \qquad V_n^2 = \sum_{i=1}^n X_i^2, \quad n \ge 1, \qquad (V_k^j)^2 = \sum_{i=k}^j X_i^2, \quad 1 \le k \le j \le n.$$

It is well known that the central limit theorem (CLT) holds, i.e., there are constants A_n and $B_n > 0$ so that, as $n \to \infty$, we have

$$\frac{S_n - A_n}{B_n} \xrightarrow{D} N(0, 1), \tag{1.1}$$

if and only if $EX^2 I(|X| \le x)$ is a slowly varying function as $x \to \infty$. This is one of the necessary and sufficient analytic conditions (cf. Theorem 1a in [1, page 313]) for X to be in the domain of attraction of the normal law, written $X \in$ DAN. It is known that A_n can be taken as nEX and $B_n = n^{1/2}\ell_X(n)$ with some function $\ell_X(n)$ that is slowly varying at infinity, defined by the distribution of X. Moreover $\ell_X(n) = \sqrt{\operatorname{Var} X} > 0$, if $\operatorname{Var} X < \infty$, and $\ell_X(n) \uparrow \infty$, as $n \to \infty$, if $\operatorname{Var} X = \infty$. Also, X has moments of all orders less than 2, and the variance of X is positive, but need not be finite.

It has become well established in the past twenty or so years that limit theorems for self-normalized sums S_n/V_n require fewer, frequently much fewer, moment assumptions than those that are necessary for their classical analogues. Consequently, the asymptotic theory of self-normalized sums has much extended the scope of the classical theory. For a global overview of these developments we refer to the papers by Shao [2–4], Csörgő et al. [5], Jing et al. [6,7] and Zhou and Jing [8] and to the book by de la Peña et al. [9]. Here we only mention those developments that have led us to posing and proving Theorem 1.1, the main result of this exposition.

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To begin with, in contrast to the well-known Hartman–Wintner law of the iterated logarithm and its converse by Strassen [10], in their seminal work Griffin and Kuelbs [11] obtained a self-normalized law of the iterated logarithm (LIL) for all distributions in the domain of attraction of a stable law that in the case of $X \in$ DAN and $EX = 0$ reads

$$\limsup_{n\to\infty} \max_{1\le i\le n} \frac{|S_i|}{V_n(2\log\log n)^{1/2}} = 1 \quad \text{a.s.} \tag{1.2}$$

When $\sigma^2 := EX^2 < \infty$, then, on account of Kolmogorov's law of large numbers (1.2) reduces to the classical Hartman–Wintner [12] LIL, which holds if and only if EX^2 is finite (cf. [10]). The general sample path properties of increments of scalar normalized partial sums also depend on, and are characterized by, moment conditions. For example, a result of Csörgő and Révész [13,14] for such increments via Shao [15] says that if $\{a_n,\ n \ge 1\}$ is a non-decreasing sequence of positive numbers satisfying

(i) $1 \le a_n \le n$,
(ii) $n^{-1}a_n$ is non-increasing,
(iii) $a_n/\log n \to \infty$ as $n \to \infty$,

then we have

$$\limsup_{n\to\infty} \max_{0\le k\le n} \max_{1\le i\le a_n} \frac{|S_{k+i} - S_k|}{(2a_n(\log(n/a_n) + \log\log n))^{1/2}} = 1 \quad \text{a.s.} \tag{1.3}$$

if and only if

$$EX = 0, \quad EX^2 = 1 \quad \text{and} \quad Ee^{t_0|X|} < \infty \quad \text{for some } t_0 > 0. \tag{1.4}$$

Motivated by the self-normalized LIL of Griffin and Kuelbs [11], Csörgő et al. [16] obtained an analogue for self-normalized increments of partial sums, which reads as follows (cf. Theorem 7 in [2]).

Theorem A. *If $X \in$ DAN and $EX = 0$, then*

$$\limsup_{n\to\infty} \max_{0\le k\le n} \max_{1\le i\le a_n} \frac{|S_{k+i} - S_k|}{\left(\sum_{k<j\le k+a_n} X_j^2\right)^{1/2} (2(\log(n/a_n) + \log\log n))^{1/2}} = 1 \quad a.s. \tag{1.5}$$

for a non-decreasing sequence of positive numbers $\{a_n,\ n \ge 1\}$ that satisfy (i)–(iii) *above.*

Consequently, on letting $k = 0$ and $a_n = n$, (1.5) yields (1.2).

We call attention to the fact that the strong moment conditions in (1.4) that are also necessary for having the scalar scaled strong result for increments of partial sums in (1.3) are replaced by the much weaker $X \in$ DAN and $EX = 0$ assumptions for having (1.5), the self-normalized version of (1.3).

Let again $\{a_n,\ n \ge 1\}$ be a non-decreasing sequence of positive integers and define $\Gamma_n(\cdot)$, a sequence of LIL-adapted self-normalized increments of partial sum processes in $C[0, 1]$, as follows:

$$\left\{\Gamma_n(t),\ 0 \le t \le 1\right\}_{n\ge 3} = \left\{\frac{S_{n-a_n+[a_nt]} - S_{n-a_n} + (a_nt - [a_nt])X_{n-a_n+[a_nt]+1}}{V^n_{n-a_n+1}\beta_n},\ 0 \le t \le 1\right\}_{n\ge 3}, \tag{1.6}$$

where $\beta_n = \sqrt{2(\log(na_n^{-1}) + \log\log n)}$.

The main result of this paper reads as follows.

Theorem 1.1. *Suppose that $EX = 0$ and $EX^2I(|X| \le x)$ is slowly varying as $x \to \infty$. Let $\{a_n,\ n \ge 1\}$ be a non-decreasing sequence of positive integers satisfying*

(i) $1 \le a_n \le n$,
(ii) *$n^{-1}a_n$ is non-increasing,*
(iii) *$a_n/\log n \to \infty$ as $n \to \infty$.*

Then, as $n \to \infty$, the sequence of random functions $\{\Gamma_n(\cdot)\}_{n\ge 3}$ is almost surely relatively compact in $C[0, 1]$ in the uniform topology and the set of its limit points coincides with $\mathcal{K}$, the class of absolutely continuous functions on $[0, 1]$ for which

$$f(0) = 0 \quad \text{and} \quad \int_0^1 (f'(x))^2 dx \le 1.$$

We note that $\mathcal{K}$ in our Theorem 1.1 is of course the very Strassen [17] class of functions that first appeared in his famous conclusion that, with a standard Brownian motion $\{W(s), 0 \le s < \infty\}$, the sequence of random elements $\{W(nt)/(2n\log\log n)^{1/2},\ 0 \le t \le 1\}_{n\ge 3}$ is, as $n \to \infty$, almost surely relatively compact in $C[0, 1]$ in the uniform topology, and the set of its limit points is $\mathcal{K}$.