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Chaoticity and invariant measures for a cell population model

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ABSTRACT

We present a structured model of a cell reproduction system given by a partial differential equations with a nonlocal division term. This equation generates semiflows acting on some subspaces of locally integrable functions. We show that these semiflows possess invariant mixing measures positive on open sets. From this it follows that the system is chaotic, i.e., it has dense trajectories and each trajectory is unstable. We also show the chaoticity of this system in the sense of Devaney.

1. Introduction

This paper is devoted to study the existence of invariant mixing measures for some semiflows generated by a partial differential equation of population dynamics. If a semiflow has an invariant measure having strong ergodic properties, then it is also chaotic. Although the idea of applying an ergodic theory approach to study chaos is rather old (see [1–3] and the monograph [4]), there are only a few examples concerning semiflows generated by partial differential equations. The problem is that such semiflows are defined on the spaces of functions and it is not easy to study evolution of measures under the action of infinite dimensional semiflows.

In this paper we consider the following equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(gxu) = -(m+d)u(t,x) + 4du(t,2x), \tag{1}$$

with constant g, m, d, and with the initial condition

$$u(0,x) = u_0(x), \quad x \in [0,\infty). \tag{2}$$

Here x is the size of a cell and $u(t,x)$ describes the density function of cells with respect to their size, i.e., $\int_0^m u(t,x)\,dx$ is the number (or biomass) of cells with size $\le m$ at time t. We do not assume here that $u(t,\cdot)$ is a probability density function and $\int_0^\infty u(t,x)\,dx$ can change in time (and even this integral can be infinite). We assume that the cell grows according to the equation

$$x'(t) = gx(t).$$

Constants m and d are, respectively, mortality and division rates. It means that in the time interval of the length Δt, a cell with size x can die with probability $m\Delta t$ and split with probability $d\Delta t$ into two cells with sizes $x/2$.

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Eq. (1) is a special case of the size-structured model

$$\frac{\partial}{\partial t}u(t,x)+\frac{\partial}{\partial x}(g(x)u(t,x))=-(m(x)+d(x))u(t,x)+4d(2x)u(t,2x) \tag{3}$$

which was introduced by Bell and Anderson [5] and was studied and generalised in many papers (see e.g. [6–10]). Usually, size x is a number from the interval $[0, 1]$ and in order to have size less than 1, it is assumed that

$$\int_0^1 b(x)\,dx=\infty. \tag{4}$$

If $g(2x)\neq 2g(x)$ at least for one $x\in[0,1]$, then the solutions of (3) have asynchronous exponential growth, i.e., there exist $\lambda\in\mathbb{R}$ and positive functions f_* and w such that

$$e^{-\lambda t}u(t,\cdot)\to f_*\int_a^1 u(0,x)w(x)\,dx \quad \text{in } L^1[0,1]$$

(see [6,9]). Having in mind this result, it is difficult to imagine that some versions of this model can be chaotic. The first results in this direction were obtained by Howard [11] and El Mourchid et al. [12]. In [12] the authors consider Eq. (3) with $g(x)=gx$, $m(x)=m$, $d(x)=d$. In order to have size less than 1 they multiply $u(t,2x)$ by $\mathbf{1}_{[0,1/2]}(x)$. Though their result concerning chaoticity of this equation is very interesting, the biological model has a disadvantage. Cells can achieve size greater than one, but they cannot replicate (larger cells may only grow and die). In our model we eliminate this problem assuming that the size x can be any positive number. It seems that, on the contrary, in this model large cells can divide.

We can consider Eq. (1) as an evolution equation on some space X of initial functions, e.g., $L^1[0,\infty)$. It is not difficult to check, however, that if $X=L^1[0,\infty)$, then the solutions of (1) do not behave in a chaotic way. We extend the space $L^1[0,\infty)$ to some subspace X of the space of locally integrable functions and we show that Eq. (1) can generate a chaotic semiflow $\{U_t\}_{t\geq 0}$ on X. We prove a strong result concerning the existence of an invariant and mixing measure with respect to $\{U_t\}_{t\geq 0}$ supported on the whole space X. The existence of such an invariant measure implies chaoticity in the sense of Auslander and Yorke [13] and the existence of turbulent solutions in the sense of Bass (the definitions are recalled in Section 5). The idea of the proof of the main results is ideologically simple but technically difficult. We want to show that the flow $\{U_t\}_{t\geq 0}$ is isomorphic to a shift flow $S_tf(x)=f(x+t)$ on a properly chosen space of functions Y. This method was used to study semiflows generated by partial differential equations of population dynamics but without the replication term, e.g. [14,15], see also a review [16]. Although it is natural to expect that semiflows generated by first order partial differential equations can be isomorphic to the shift semiflow $\{U_t\}_{t\geq 0}$, the similar property for an equation with the replication term seems to be unexpected. The second step is to construct a mixing and invariant measure m supported on the space Y. We can do it, if we find a Gaussian process ξ_x with trajectories from the space Y. Then the measure m of a Borel subset A of Y is the probability that trajectories of ξ_x are from the set A.

It should be noted that most of the recent papers concerning chaos for semigroups of operators are based on studying spectral properties of their infinitesimal generators [17,18]. This method was used in the paper [12] and it seems to be easier than ours. But, in our opinion, the approach based on the isomorphism with shift semigroups and using invariant measures reveals why the semiflow $\{U_t\}_{t\geq 0}$ is chaotic. The second advantage of the ergodic theory approach is that we can prove much stronger results concerning chaos (see Remark 1).

2. Main results

If we substitute $u(t,x)=\bar{u}(gt,x)$ in (1) we obtain the equation

$$\frac{\partial\bar{u}}{\partial t}+x\frac{\partial\bar{u}}{\partial x}=-\frac{(m+d)}{g}\bar{u}(t,x)+\frac{4d}{g}\bar{u}(t,2x), \tag{5}$$

and finally setting $a=-(m+d+g)/g$ and $b=4d/g$ and writing u instead of $\bar{u}$ we obtain our main equation

$$\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial x}=au(t,x)+bu(t,2x). \tag{6}$$

It is not difficult to check that the solution of (6) with the initial condition $u(0,x)=u_0(x)$ is given by the formula

$$u(t,x)=e^{at}\sum_{n=0}^{\infty}\frac{(bt)^n}{n!}u_0(2^ne^{-t}x). \tag{7}$$

We shall choose some subspace X of the space of measurable functions such that for each $u_0\in X$ the formula (7) is well defined. A natural candidate for X can be the space $L^1[0,\infty)$. But since

$$\int_0^{\infty}u(t,x)\,dx=e^{(a+1+b/2)t}\int_0^{\infty}u_0(x)\,dx,$$