J. Math. Anal. Appl. 393 (2012) 177–184

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On the structure of fixed-point sets of asymptotically regular semigroups

Andrzej Wiśnicki

Andrzej Wiśnicki, Institute of Mathematics, Maria Curie-Skłodowska University, 20-031 Lublin, Poland

ARTICLE INFO

Article history:
Received 14 January 2012
Available online 13 April 2012
Submitted by T.D. Benavides

Keywords:
Asymptotically regular mapping
Retraction
Fixed point
Opial property
Bynum's coefficients
Weakly null nets

ABSTRACT

We show that the set of fixed points of an asymptotically regular mapping acting on a convex and weakly compact subset of a Banach space is, in some cases, a Hölder continuous retract of its domain. Our results qualitatively complement the corresponding fixed point existence theorems and extend a few recent results of Górnicki [15–17]. We also characterize Bynum's coefficients and the Opial modulus in terms of nets.

1. Introduction

The notion of asymptotic regularity, introduced by Browder and Petryshyn in [1], has become a standing assumption in many results concerning fixed points of nonexpansive and more general mappings. Recall that a mapping $T : M \to M$ acting on a metric space (M, d) is said to be asymptotically regular if

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$$

for all $x \in M$. Ishikawa [2] proved that if C is a bounded closed convex subset of a Banach space X and $T : C \to C$ is nonexpansive, then the mapping $T_\lambda = (1-\lambda)I + \lambda T$ is asymptotically regular for each $\lambda \in (0, 1)$. Edelstein and O'Brien [3] showed independently that T_λ is uniformly asymptotically regular over $x \in C$, and Goebel and Kirk [4] proved that the convergence is even uniform with respect to all nonexpansive mappings from C into C. Other examples of asymptotically regular mappings are given by the result of Anzai and Ishikawa [5] (see also [6]): if T is an affine mapping acting on a bounded closed convex subset of a locally convex space X, then $T_\lambda = (1-\lambda)I + \lambda T$ is uniformly asymptotically regular.

In 1987, Lin [7] constructed a uniformly asymptotically regular Lipschitz mapping in ℓ_2 without fixed points which extended an earlier construction of Tingley [8]. Subsequently, Maluta et al. [9] proved that there exists a continuous fixed-point free asymptotically regular mapping defined on any bounded convex subset of a normed space which is not totally bounded (see also [10]). For the fixed-point existence theorems for asymptotically regular mappings we refer the reader to [11–13].

It was shown in [14] that the set of fixed points of a k-uniformly Lipschitzian mapping in a uniformly convex space is a retract of its domain if k is close to 1. In recent papers [15–17], Górnicki proved several results concerning the structure of fixed-point sets of asymptotically regular mappings in uniformly convex spaces. In this paper we continue this work and extend a few results of Górnicki in two aspects: we consider a more general class of spaces and prove that in some cases, the fixed-point set Fix T is not only a (continuous) retract but even a Hölder continuous retract of the domain. We present our results in a more general case of a one-parameter nonlinear semigroup. We also characterize Bynum's coefficients and the Opial modulus in terms of nets.

E-mail address: awisnic@hektor.umcs.lublin.pl.

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doi:10.1016/j.jmaa.2012.03.036

2. Preliminaries

Let G be an unbounded subset of $[0, \infty)$ such that $t + s, t - s \in G$ for all $t, s \in G$ with $t > s$ (e.g., $G = [0, \infty)$ or $G = \mathbb{N}$). By a nonlinear semigroup on C we shall mean a one-parameter family of mappings $\mathcal{T} = \{T_t : t \in G\}$ from C into C such that $T_{t+s}x = T_t T_s x$ for all $t, s \in G$ and $x \in C$. In particular, we do not assume in this paper that $\{T_t : t \in G\}$ is strongly continuous. We use a symbol $|T|$ to denote the exact Lipschitz constant of a mapping $T : C \to C$, i.e.,

$$|T| = \inf\{k > 0 : \|Tx - Ty\| \leq k\|x - y\| \text{ for all } x, y \in C\}.$$

If T is not Lipschitzian we define $|T| = \infty$.

A semigroup $\mathcal{T} = \{T_t : t \in G\}$ from C into C is said to be asymptotically regular if $\lim_t \|T_{t+h}x - T_t x\| = 0$ for every $x \in C$ and $h \in G$.

Assume now that C is convex and weakly compact and $\mathcal{T} = \{T_t : t \in G\}$ is a nonlinear semigroup on C such that $s(\mathcal{T}) = \liminf_t |T_t| < \infty$. Choose a sequence (t_n) of elements in G such that $\lim_{n\to\infty} t_n = \infty$ and $s(\mathcal{T}) = \lim_{n\to\infty} |T_{t_n}|$. By Tikhonov's theorem, there exists a pointwise weakly convergent subnet $(T_{t_{n_\alpha}})_{\alpha\in A}$ of (T_{t_n}). We denote it briefly by $(T_{t_\alpha})_{\alpha\in A}$. For every $x \in C$, define

$$Lx = w\text{-}\lim_\alpha T_{t_\alpha} x, \tag{1}$$

i.e., Lx is the weak limit of the net $(T_{t_\alpha}x)_{\alpha\in A}$. Notice that Lx belongs to C since C is convex and weakly compact. The weak lower semicontinuity of the norm implies

$$\|Lx - Ly\| \leq \liminf_\alpha \|T_{t_\alpha}x - T_{t_\alpha}y\| \leq \limsup_{n\to\infty} \|T_{t_n}x - T_{t_n}y\| \leq s(\mathcal{T})\|x - y\|.$$

We formulate the above observation as a separate lemma.

Lemma 2.1. *Let C be a convex weakly compact subset of a Banach space X and let $\mathcal{T} = \{T_t : t \in G\}$ be a semigroup on C such that $s(\mathcal{T}) = \liminf_t |T_t| < \infty$. Then the mapping $L : C \to C$ defined by (1) is $s(\mathcal{T})$-Lipschitz.*

We end this section with the following variant of a well known result which is crucial for our work (see, e.g., [18, Proposition 1.10]).

Lemma 2.2. *Let (X, d) be a complete bounded metric space and let $L : X \to X$ be a k-Lipschitz mapping. Suppose there exist $0 < \gamma < 1$ and $c > 0$ such that $d(L^{n+1}x, L^n x) \leq c\gamma^n$ for every $x \in X$. Then $Rx = \lim_{n\to\infty} L^n x$ is a Hölder continuous mapping.*

Proof. We may assume that $\operatorname{diam} X = 1$. Fix $x \neq y$ in X and notice that for any $n \in \mathbb{N}$,

$$d(Rx, Ry) \leq d(Rx, L^n x) + d(L^n x, L^n y) + d(L^n y, Ry) \leq 2c\frac{\gamma^n}{1-\gamma} + k^n d(x, y).$$

Take $\alpha < 1$ such that $k \leq \gamma^{1-\alpha^{-1}}$ and put $\gamma^{n-r} = d(x, y)^\alpha$ for some $n \in \mathbb{N}$ and $0 < r \leq 1$. Then $k^{n-1} \leq (\gamma^{1-\alpha^{-1}})^{n-r}$ and hence

$$d(Rx, Ry) \leq 2c\frac{\gamma^{n-r}}{1-\gamma} + k(\gamma^{n-r})^{1-\alpha^{-1}} d(x, y) = \left(\frac{2c}{1-\gamma} + k\right) d(x, y)^\alpha. \quad \square$$

3. Bynum's coefficients and the Opial modulus in terms of nets

From now on, C denotes a nonempty convex weakly compact subset of a Banach space X. Let $\mathcal{A}$ be a directed set, $(x_\alpha)_{\alpha\in\mathcal{A}}$ a bounded net in X, $y \in X$ and write

$$r(y, (x_\alpha)) = \limsup_\alpha \|x_\alpha - y\|,$$

$$r(C, (x_\alpha)) = \inf\{r(y, (x_\alpha)) : y \in C\},$$

$$A(C, (x_\alpha)) = \{y \in C : r(y, (x_\alpha)) = r(C, (x_\alpha))\}.$$

The number $r(C, (x_\alpha))$ and the set $A(C, (x_\alpha))$ are called, respectively, the asymptotic radius and the asymptotic center of $(x_\alpha)_{\alpha\in\mathcal{A}}$ relative to C. Notice that $A(C, (x_\alpha))$ is nonempty convex and weakly compact. Write

$$r_a(x_\alpha) = \inf\{\limsup_\alpha \|x_\alpha - y\| : y \in \overline{\operatorname{conv}}(\{x_\alpha : \alpha \in \mathcal{A}\})\}$$

and let

$$\operatorname{diam}_a(x_\alpha) = \inf_\alpha \sup_{\beta,\gamma\geq\alpha} \|x_\beta - x_\gamma\|$$

denote the asymptotic diameter of (x_α).