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L^∞ a priori bounds for gradients of solutions to quasilinear inhomogeneous fast-growing parabolic systems

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ABSTRACT

We prove boundedness of gradients of solutions to quasilinear parabolic systems, the main part of which is a generalization to the p-Laplacian and its right-hand side's growth depending on the gradient is not slower (and generally strictly faster) than $p-1$. This result may be seen as a generalization to the classical notion of a controllable growth of the right-hand side, introduced by Campanato, over gradients of p-Laplacian-like systems. Energy estimates and a nonlinear iteration procedure of the Moser type are cornerstones of the used method.

1. Introduction

1.1. General statement of the problem

We are interested in obtaining a local boundedness of gradients of solutions to the following parabolic system in $\Omega \subset \mathbb{R}^n$

$$u^i_t - (A^i_\alpha(\nabla u))_{x_\alpha} = f^i(x, t, \nabla u) \quad i = 1, \dots, N$$

where the main part is a generalization of the p-Laplacian and the right-hand side grows as $1 + |\nabla u|^w$ or $|\nabla u|^w$ with w specified further. We say that a right-hand side is a fast-growing one, when $w > p - 1$ holds.

The existing literature on the regularity issue of parabolic equations and systems is impressive. Let us recall that for equations the existing results are quite strong: even for the right-hand-side growth of $1+|\nabla u|^p$ one obtains $C^{1,\alpha}$ regularity of solutions: see classic monograph Ladyzhenskaya et al. [1] for the case $p = 2$ and DiBenedetto [2] for $p \in (1, \infty)$. Many further generalizations are possible: for instance in [3] the right-hand side takes the form $e^u|\nabla u|^p$, which suffices for a boundedness of ∇u. Moreover, this growth condition seems to be optimal, because there are blow-up results for gradients of solutions to equations, the right-hand sides of which grow faster than p—compare Souplet [4]. In the case of systems, the regularity results are much weaker. One can construct irregular (i.e. unbounded or discontinuous) functions, which solve homogeneous parabolic systems. For $n > 2$ it suffices for irregularity that the coefficients $A(x, t)$ of the main part are discontinuous (and still bounded) or that there is a relevant non-diagonality of the main part—for details, consult Arkhipova [5]. Nevertheless, there are many classes of main part which allow for higher regularity (even $C^{1,\alpha}$) in the homogeneous case; these are: having structure close to Laplacian or p-Laplacian, like those studied in [1] or [2],

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respectively, or having main part depending solely on ∇u: see the well-known paper by Nečas and Šverak [6] or more extensive research done by Choe and Bae [7]. As these papers consider homogeneous systems, one may ask a natural question: what inhomogeneous counterparts of such systems remain, in a certain sense, regular? The general answer is unknown, but there are several hints: on one hand, for the right-hand side growing like $1 + |\nabla u|^{p-1}$ the regularity of the homogeneous case seems to be retained—see [2]; on the other, unlike for equations, one cannot have the right-hand side growing as fast as $|\nabla u|^p$ without further assumptions, even in the case of a system with the simplest main part, i.e. an inhomogeneous heat system. Recall the classical counterexample: for $n \geq 3$ bounded but discontinuous function $u(x) = \frac{x}{|x|}$ with unbounded weak derivatives solves $u^i_t - \Delta u^i = u^i |\nabla u|^2 \left[= (n-1)\frac{x^i}{|x|^3}\right]$, for details—see [8]. It turns out that in the case of an inhomogeneous system for $p = 2$ that one has to additionally assume a certain smallness in order to obtain regularity—for details, refer to Tolksdorf [9], Pingen [10], Idone [11] or even the classical Ladyzhenskaya et al. [1]. The regularity issue for a general nonlinear inhomogeneous parabolic system with the right-hand side growing at the rate $1 + |\nabla u|^w$ for w possibly close to p, homogeneous counterparts of which enjoy regularity, is not fully researched, especially for the case $p \neq 2$. There are several approaches to answering this question: some authors relax the notion of regularity by resorting to partial regularity—see for example classical papers of Italian school: Campanato [12], Giaquinta and Struwe [13] and newer ones: Fanciullo [14], Frehse and Specovius–Neugebauer [15], Misawa [16], Duzaar and Mingione [17]; or by demanding a high integrability-type regularity[2], like in [18,19] or [20] (in the last paper the growth of the right-hand-side may be polynomially arbitrarily large!). Certain systems with peculiar structure or two-dimensional ones (or at least close to them in some sense) enjoy also high regularity, even if they are much more general than a Stokes-type system; for results in this direction compare the papers of Seregin, Arkhipova, Frehse, Kaplicky (and many others), for instance: Arkhipova [5], Naumann and Wolff [21], Kaplicky [22], Zajączkowski and Seregin [23].

In this note we focus on deriving a full regularity result, more precisely: the local boundedness of gradients, for a class of quasilinear parabolic inhomogeneous systems. Our goal is twofold: firstly to obtain results for a general inhomogeneous parabolic system, the main part of which is analogous to the system considered in [7], while retaining possibly general growth conditions for the right-hand side. Secondly, to sharpen these results with respect to growth of the right-hand side, restricting ourselves to less general systems, being close to p-Laplacian. For similar result on the level of solutions, compare Giorgi and O'Leary [24].

Let us emphasize that we proceed in a manner typical for the regularity approach: we assume existence of solution u in a given class, which is often a deep problem itself, from which we derive higher regularity. Moreover, we concentrate on a priori estimates while conducting the proofs: the rigorous version of computations is commented on in the conclusion.

1.2. General definitions and assumptions

Consider the parabolic problem in $\Omega \subset \mathbb{R}^n$

$$u^i_t - (A^i_\alpha(\nabla u))_{x_\alpha} = f^i(x, t, \nabla u) \quad i = 1, \dots, N. \tag{1}$$

As all our results have a local character, any further specification of Ω is irrelevant.

We say that a vector valued function $u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$ is a weak solution to (1) iff

$$\int_{\Omega_T} -u^i\phi^i_t + A^i_\alpha(\nabla u)\phi^i_{x_\alpha} dxdt = \int_{\Omega_T} f^i(x, t, \nabla u)\phi^i dxdt \qquad \forall_{\phi \in C_0^\infty(\Omega_T)} .$$

Globally, the following notions will be used:

- δ^α_β denotes the Kronecker delta,
- $Q_R(x_0, t_0)$ denotes a parabolic cylinder, i.e. $B_R(x_0) \times (t_0 - R^p, t_0)$; when possible, cut short to Q_R,
- $\eta_{\rho,R} \in C_0^\infty(Q_R)$ denotes a standard parabolic cutoff function for $Q_\rho \subset Q_R$, when possible, cut short to η, which satisfies $\eta = 1$ in Q_ρ, $\eta = 0$ outside Q_R, $|\nabla \eta| \leq c(R-\rho)^{-1}$, $|\eta_{,t}| \leq c(R-\rho)^{-p}$.

Throughout the article, summation over repeated indices is in use.

1.3. The structure of results

We show our results in the following order:

1. First, we derive a general result for inhomogeneous version of system analyzed in [7], where ellipticity assumptions for the main part are generalized by introducing exponent q (Theorem 2). Here, loosely speaking, admissible growth for the right-hand side is $1 + |\nabla u|^{p-1}$, so this result may be seen as parallel to DiBenedetto [2].

[2] Such results are especially interesting, as our result may be easily strengthened via higher regularity