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Regularity and symmetry for solutions to a system of weighted integral equations

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ABSTRACT

We study positive solutions of the following system of integral equations related to the weighted Hardy–Littlewood–Sobolev inequality:

$$\begin{cases} u(x) = \dfrac{1}{|x|^{\alpha}} \displaystyle\int_{R^n} \dfrac{v^q(y)}{|y|^{\beta}|x-y|^{\lambda}}\,dy, \\ v(x) = \dfrac{1}{|x|^{\beta}} \displaystyle\int_{R^n} \dfrac{u^p(y)}{|y|^{\alpha}|x-y|^{\lambda}}\,dy, \end{cases}$$

where $u, v \geqslant 0$, $0 < p, q < \infty$, $\alpha, \beta \geqslant 0$, $0 < \lambda < n$, $\frac{\alpha}{n} < \frac{1}{p+1} < \frac{\lambda+\alpha}{n}$ and $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}$. We obtain regularity of the solutions by regularity-lifting-method, which has been extensively used by many authors. Moreover, using the method of moving planes in integral forms, we also establish symmetry of the solutions under the weaker condition that u and v are only locally integrable.

1. Introduction

Let $0 < \lambda < n$ and let $1 < s, r < \infty$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2$. Let $\|f\|_p$ be the $L^p(R^n)$ norm of the function f. The well-known classical Hardy–Littlewood–Sobolev inequality (HLS) states that

$$\int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x-y|^{\lambda}}\,dx\,dy \leqslant C_{s,\lambda,n}\|f\|_r\|g\|_s \tag{1}$$

for any $f \in L^r(R^n)$ and $g \in L^s(R^n)$.

Hardy and Littlewood also introduced the double weighted inequality, which was generalized by Stein and Weiss in [30]. This inequality is called double weighted Hardy–Littlewood–Sobolev (WHLS) inequality:

$$\int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}\,dx\,dy \leqslant C_{\alpha,\beta,s,\lambda,n}\|f\|_r\|g\|_s \tag{2}$$

where $1 < s, r < \infty$, $0 < \lambda < n$, $\alpha + \beta \geqslant 0$ and the powers α, β of the weights satisfy

$$1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r}, \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{n} = 2. \tag{3}$$

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To find the best constant $C = C(\alpha, \beta, s, \lambda, n)$ in the weighted inequality (2), one can maximize the functional

$$J(f,g) = \int_{R^n}\int_{R^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta}\,dx\,dy \tag{4}$$

under the constrains $\|f\|_r = \|g\|_s = 1$. The corresponding Euler–Lagrange equations are the following system of integral equations:

$$\begin{cases} \lambda_1 r f(x)^{r-1} = \dfrac{1}{|x|^\alpha}\displaystyle\int_{R^n} \frac{g(y)}{|y|^\beta |x-y|^\lambda}\,dy, \\ \lambda_2 s g(x)^{s-1} = \dfrac{1}{|x|^\beta}\displaystyle\int_{R^n} \frac{f(y)}{|y|^\alpha |x-y|^\lambda}\,dy \end{cases} \tag{5}$$

where $f, g \geqslant 0$, $x \in R^n$ and $\lambda_1 r = \lambda_2 s = J(f,g)$.

Letting $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $p = \frac{1}{r-1}$, $q = \frac{1}{s-1}$, when $pq \neq 1$, and by a proper choice of constants c_1 and c_2, system (5) becomes

$$\begin{cases} u(x) = \dfrac{1}{|x|^\alpha}\displaystyle\int_{R^n} \frac{v^q(y)}{|y|^\beta |x-y|^\lambda}\,dy, \\ v(x) = \dfrac{1}{|x|^\beta}\displaystyle\int_{R^n} \frac{u^p(y)}{|y|^\alpha |x-y|^\lambda}\,dy \end{cases} \tag{6}$$

where $u, v \geqslant 0$, $0 < p, q < \infty$, $0 < \lambda < n$, $\frac{\alpha}{n} < \frac{1}{p+1} < \frac{\lambda+\alpha}{n}$, and $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}$.

In the special case, where $\alpha = 0$ and $\beta = 0$, system (6) reduces to

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \frac{v^q(y)}{|x-y|^\lambda}\,dy, \\ v(x) = \displaystyle\int_{R^n} \frac{u^p(y)}{|x-y|^\lambda}\,dy \end{cases} \tag{7}$$

with

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda}{n}.$$

It was shown in [9] that the integral system (7) is equivalent to the system of partial differential equations

$$\begin{cases} (-\Delta)^{\gamma/2} u = v^q, \quad u > 0, \text{ in } R^n, \\ (-\Delta)^{\gamma/2} v = u^p, \quad v > 0, \text{ in } R^n \end{cases} \tag{8}$$

where $\gamma = n - \lambda$.

In the special case where $p = q = \frac{n+\gamma}{n-\gamma}$, and $u(x) = v(x)$, system (7) becomes

$$u(x) = \int_{R^n} \frac{u(y)^{\frac{n+\gamma}{n-\gamma}}}{|x-y|^{n-\gamma}}\,dy, \quad u > 0, \text{ in } R^n. \tag{9}$$

The equivalent PDE is the well-known family of semi-linear equations

$$(-\Delta)^{\gamma/2} u = u^{\frac{n+\gamma}{n-\gamma}}, \quad u > 0, \text{ in } R^n. \tag{10}$$

In particular, when $n > 3$, and $\gamma = 2$, (10) becomes

$$-\Delta u = u^{(n-2)/n-2}, \quad u > 0, \text{ in } R^n. \tag{11}$$

The classification of the solutions of (11) has provided an important ingredient in the study of the well-known Yamabe problem and the prescribing scalar curvature problem. It is also essential in deriving a priori estimate in many related nonlinear elliptic equations.

Solutions to (11) were studied by Gidas, Ni, and Nirenberg [15]. They proved that all the positive solutions of (11) with reasonable behavior at infinity, namely