J. Math. Anal. Appl. 390 (2012) 596–602

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note

Perturbational blowup solutions to the 2-component Camassa–Holm equations

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ARTICLE INFO

Article history:
Received 10 March 2011
Available online 11 May 2011
Submitted by J. Xiao

Keywords:
Camassa–Holm equations
Perturbational method
Non-radial symmetry
Construction of solutions
Functional differential equations
Dynamical system
Global existence
Blowup
Radial symmetry
Emden equation

ABSTRACT

In this article, we study the perturbational method to construct the non-radially symmetric solutions of the compressible 2-component Camassa–Holm equations. In detail, we first combine the substitutional method and the separation method to construct a new class of analytical solutions for that system. In fact, we perturb the linear velocity:

$$u = c(t)x + b(t), \tag{1}$$

and substitute it into the system. Then, by comparing the coefficients of the polynomial, we can deduce the functional differential equations involving $(c(t), b(t), \rho^2(0,t))$. Additionally, we could apply Hubble's transformation $c(t) = \frac{\dot{a}(3t)}{a(3t)}$, to simplify the ordinary differential system involving $(a(3t), b(t), \rho^2(0,t))$. After proving the global or local existences of the corresponding dynamical system, a new class of analytical solutions is shown. To determine that the solutions exist globally or blow up, we just use the qualitative properties about the well-known Emden equation. Our solutions obtained by the perturbational method, fully cover Yuen's solutions by the separation method.

1. Introduction

The 2-component Camassa–Holm equations of shallow water system can be expressed by

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0, \quad x \in R, \\ m_t + 2u_x m + u m_x + \sigma\rho\rho_x = 0 \end{cases} \tag{2}$$

with

$$m = u - \alpha^2 u_{xx}. \tag{3}$$

Here $u = u(x,t) \in R$ and $\rho = \rho(x,t) \geqslant 0$ are the velocity and the density of fluid respectively. The constant σ is equal to 1 or -1. If $\sigma = -1$, the gravity acceleration points upwards [2,3,8,10,9]. For $\sigma = 1$, the researches regarding the corresponding models could be referred to [4,6,10,8]. When $\rho \equiv 0$, the system returns to the Camassa–Holm equation [1]. The searching of the Camassa–Holm equation can capture breaking waves. Peaked traveling waves is a long-standing open problem [18].

In 2010, Yuen used the separation method to obtain a class of blowup or global solutions of the Camassa–Holm equations [23] and the Degasperis–Procesi equations [24]. In particular, for the integrable system of the Camassa–Holm equations with $\sigma = 1$, we have the global solutions:

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.05.016

$$\begin{cases} \rho(x,t)=\max\left\{\dfrac{f(\eta)}{a(3t)^{1/3}},0\right\}, \quad u(x,t)=\dfrac{\dot{a}(3t)}{a(3t)}x, \\ \ddot{a}(s)-\dfrac{\xi}{3a(s)^{1/3}}=0, \quad a(0)=a_0>0, \quad \dot{a}(0)=a_1, \\ f(\eta)=\xi\sqrt{-\dfrac{\eta^2}{\xi}+(\xi\alpha)^2} \end{cases} \tag{4}$$

where $\eta=\frac{x}{a(s)^{1/3}}$ with $s=3t$; $\xi>0$ and $\alpha\geqslant 0$ are arbitrary constants [23].

Meanwhile, the isentropic compressible Euler equations can be written in the following form:

$$\begin{cases} \rho_t+\nabla\cdot\rho u=0, \\ (\rho u)_t+\nabla\cdot(\rho u\otimes u)+\nabla P=0. \end{cases} \tag{5}$$

As usual, $\rho=\rho(x,t)$ and $u=u(x,t)\in\mathbf{R}^N$ are the density and the velocity respectively with $x=(x_1,x_2,\ldots,x_N)\in R^N$. For some fixed $K>0$, we have a γ-law on the pressure:

$$P=P(\rho)=K\rho^{\gamma} \tag{6}$$

with a constant $\gamma\geqslant 1$. For solutions in radially symmetry:

$$\rho(x,t)=\rho(r,t) \quad \text{and} \quad u(x,t)=\frac{x}{r}V(r,t)=:\frac{x}{r}V \tag{7}$$

with the radial $r=\sum_{i=1}^{N}x_i^2$, the compressible Euler equations (5) become

$$\begin{cases} \rho_t+V\rho_r+\rho V_r+\dfrac{N}{r}\rho V=0, \\ \rho(V_t+VV_r)+\dfrac{\partial}{\partial r}P=0. \end{cases} \tag{8}$$

Recently, there are some researches concerning the construction of solutions of the compressible Euler and Navier–Stokes equations by the substitutional method [12,13,19,14]. They assume that the velocity is linear:

$$u(x,t)=c(t)x \tag{9}$$

and substitute it into the system to derive the dynamic system about the function $c(t)$. Then they use the standard argument of phase diagram to derive the blowup or global existence of the ordinary differential equation involving $c(t)$.

On the other hand, the separation method can be governed to seek the radial symmetric solutions by the functional form:

$$\rho(r,t)=\frac{f(\frac{r}{a(t)})}{a^N(t)} \quad \text{and} \quad V(r,t)=\frac{\dot{a}(t)}{a(t)}r. \tag{10}$$

(See [7,15,16,5,12,19–22].)

It is natural to consider the more general linear velocity:

$$u(x,t)=c(t)x+b(t) \tag{11}$$

to construct new solutions. In this article, we can first combine the two conventional approaches (substitutional method and separation method) to derive the corresponding solutions for the system. In fact, the main theme of this article is to substitute the linear velocity (11) into the Camassa–Holm equations (2) and compare the coefficient of the different polynomial degrees for deducing the functional differential equations involving $(c(t),b(t),\rho^2(0,t))$. Then, we can apply Hubble's transformation

$$c(t)=\frac{\dot{a}(3t)}{a(3t)} \tag{12}$$

with $\dot{a}(3t):=\frac{da(3t)}{dt}$ to simplify the functional differential system involving $(a(3t),b(t),\rho^2(0,t))$. After proving the local existences of the corresponding dynamical system, we can show the results below: