J. Math. Anal. Appl. 390 (2012) 1–14

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Convergence rates in precise asymptotics

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ARTICLE INFO

Article history:
Received 20 May 2011
Available online 29 November 2011
Submitted by V. Pozdnyakov

Keywords:
Law of large numbers
Baum–Katz
Precise asymptotics
Convergence rates

ABSTRACT

Let $X_1, X_2, \ldots$ be i.i.d. random variables with partial sums S_n, $n \geqslant 1$. The now classical Baum–Katz theorem provides necessary and sufficient moment conditions for the convergence of $\sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| \geqslant \varepsilon n^{1/p})$ for *fixed* $\varepsilon > 0$. An equally classical paper by Heyde in 1975 initiated what is now called precise asymptotics, namely asymptotics for the same sum (for the case $r = 2$ and $p = 1$) when, instead, $\varepsilon \searrow 0$. In this paper we extend a result due to Klesov (1994), in which he determined the convergence rate in Heyde's theorem.

1. Introduction

In the seminal paper [11] Hsu and Robbins introduced the concept of *complete convergence*, and proved that the sequence of arithmetic means of independent, identically distributed (i.i.d.) random variables converges completely (which means that the Borel–Cantelli sum of certain tail probabilities converges) to the expected value of the variables, provided their variance is finite. The necessity was proved afterwards by Erdős [5,6]. The Hsu–Robbins–Erdős result was later extended in a series of papers which culminated in the paper by Baum and Katz [1]. The following result is a part of their main result.

Theorem 1.1. *Let $r > 0$, $0 < p < 2$ and $r \geqslant p$. Suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables with $E|X|^r < \infty$ and, if $r \geqslant 1$, $EX = 0$, and set $S_n = \sum_{k=1}^{n} X_k$, $n \geqslant 1$. Then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2} P\big(|S_n| \geqslant \varepsilon n^{1/p}\big) < \infty, \quad \text{for all } \varepsilon > 0. \tag{1.1}$$

Conversely, if the sum is finite for some $\varepsilon > 0$, then $E|X|^r < \infty$ and, if $r \geqslant 1$, $EX = 0$. In particular, the conclusion then holds for all $\varepsilon > 0$.

Remark 1.1. For $r = 2$ and $p = 1$ the result reduces to the theorem of Hsu and Robbins [11] (sufficiency) and Erdős [5,6] (necessity). For $r = p = 1$ we rediscover the famous theorem of Spitzer [17]. For $r > 0$ and $p = 1$ the result was earlier proved by Katz; see [12]. □

Results of this kind naturally provide information about the *rate* at which the probabilities in (1.1) converge to zero for fixed ε. Another problem of interest is to ask for the rate at which these probabilities tend to one as $\varepsilon \searrow 0$. Toward that end, Heyde [10] proved that

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.11.046

$$\lim_{\varepsilon\searrow 0}\varepsilon^2\sum_{n=1}^{\infty}P\big(|S_n|\geqslant\varepsilon n\big)=EX^2,\tag{1.2}$$

whenever $EX=0$ and $EX^2<\infty$. For the remaining values of r and p we refer to [3,16,8]. For ease of reference we state the main result from [3] which is relevant for our purpose here.

Theorem 1.2. *Let $r\geqslant 2$ and $0<p<2$. Suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables with $EX=0$, $EX^2=\sigma^2>0$ and $E|X|^r<\infty$, and set $S_n=\sum_{k=1}^n X_k$, $n\geqslant 1$. Then*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty}n^{(r/p)-2}P\big(|S_n|\geqslant\varepsilon n^{1/p}\big)=\frac{p}{r-p}E|Z|^{2(r-p)/(2-p)},\tag{1.3}$$

where Z is normal with mean 0 and variance $\sigma^2>0$.

Results of this kind are frequently called "Precise asymptotics for…", and an abundance of papers with various extensions of the i.i.d. case and the power weights have been produced. For an extensive review we refer [9].

The following result, due to Klesov [13], gives information about the rate of convergence in Heyde's (rate) result (1.2).

Theorem 1.3. *Let $X, X_1, X_2, \ldots$ be i.i.d. random variables, and set $S_n=\sum_{k=1}^n X_k$, $n\geqslant 1$.*

(a) *If X is normal with mean 0 and variance $\sigma^2>0$, then*

$$\lim_{\varepsilon\searrow 0}\left(\sum_{n=1}^{\infty}P\big(|S_n|\geqslant\varepsilon n\big)-\frac{\sigma^2}{\varepsilon^2}\right)=-\frac{1}{2}.$$

(b) *If $EX=0$, $EX^2=\sigma^2>0$, and $E|X|^3<\infty$, then*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{3/2}\left(\sum_{n=1}^{\infty}P\big(|S_n|\geqslant\varepsilon n\big)-\frac{\sigma^2}{\varepsilon^2}\right)=0.$$

The aim of the present paper is to prove the following extension of Klesov's theorem with respect to Theorem 1.2.

Theorem 1.4. *Let $r\geqslant 2$ and $0<p<2$. Suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables, and set $S_n=\sum_{k=1}^n X_k$, $n\geqslant 1$. Let Z be normal with mean 0 and variance $\sigma^2>0$.*

(a) *If $EX=0$, $EX^2=\sigma^2>0$, and $E|X|^q<\infty$ for some $r<q\leqslant 3$, then*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{q(r-p)/(q-p)}\left(\sum_{n=1}^{\infty}n^{(r/p)-2}P\big(|S_n|\geqslant\varepsilon n^{1/p}\big)-\frac{p}{r-p}\varepsilon^{-2(r-p)/(2-p)}E|Z|^{2(r-p)/(2-p)}\right)=0.$$

(b) *If $EX=0$, $EX^2=\sigma^2>0$, and $E|X|^q<\infty$ for some $q\geqslant 3$ with $q>(2r-3p)/(2-p)$, then*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2q(r-p)/(p+q(2-p))}\left(\sum_{n=1}^{\infty}n^{(r/p)-2}P\big(|S_n|\geqslant\varepsilon n^{1/p}\big)-\frac{p}{r-p}\varepsilon^{-2(r-p)/(2-p)}E|Z|^{2(r-p)/(2-p)}\right)=0.$$

Remark 1.2. Theorem 1.4(a)–(b) extends the above Theorem 1.3(b) of Klesov [13], since, for $r=2$, $p=1$, and $q=3$, one has $q(r-p)/(q-p)=3/2=2q(r-p)/(p+q(2-p))$. □

The proof of Theorem 1.4 is based on the following proposition concerning the Gaussian case and a Berry–Esseen type remainder term argument.

Proposition 1.1. *Let $0<p<2$ and $r\geqslant 2$, and suppose that $Z; X_1, X_2, \ldots$ are i.i.d. normal random variables with mean 0 and variance $\sigma^2>0$, and set $S_n=\sum_{k=1}^n X_k$, $n\geqslant 1$.*

(i) *If $0<r<2p$, then*

$$\lim_{\varepsilon\searrow 0}\left(\sum_{n=1}^{\infty}n^{(r/p)-2}P\big(|S_n|\geqslant\varepsilon n^{1/p}\big)-\frac{p}{r-p}\cdot\varepsilon^{-\frac{2(r-p)}{2-p}}E|Z|^{\frac{2(r-p)}{2-p}}\right)=0.$$