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A note on the absurd law of large numbers in economics

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ABSTRACT

Let Γ be a Borel probability measure on $\mathbb{R}$ and $(T, \mathcal{C}, Q)$ a nonatomic probability space. Define $\mathcal{H} = \{H \in \mathcal{C}\colon\ Q(H) > 0\}$. In some economic models, the following condition is requested. There is a probability space $(\Omega, \mathcal{A}, P)$ and a real process $X = \{X_t\colon\ t \in T\}$ satisfying

for each $H \in \mathcal{H}$, there is $A_H \in \mathcal{A}$ with $P(A_H) = 1$ such that

$t \mapsto X(t, \omega)$ is measurable and $Q(\{t\colon\ X(t,\omega) \in \cdot\} \mid H) = \Gamma(\cdot)$ for $\omega \in A_H$.

Such a condition fails if P is countably additive, $\mathcal{C}$ countably generated and Γ nontrivial. Instead, as shown in this note, it holds for any $\mathcal{C}$ and Γ under a finitely additive probability P. Also, X can be taken to have any given distribution.

1. Introduction and result

Let $(T, \mathcal{C}, Q)$ and $(\Omega, \mathcal{A}, P)$ be probability spaces and $X : T \times \Omega \to \mathbb{R}$ a real stochastic process, indexed by T and defined on $(\Omega, \mathcal{A}, P)$. Denote by $X_t(\cdot) = X(t, \cdot)$ and $X^\omega(\cdot) = X(\cdot, \omega)$ the X-sections with respect to $t \in T$ and $\omega \in \Omega$. Since X is a process, $X_t : \Omega \to \mathbb{R}$ is measurable for fixed $t \in T$.

In various economic frameworks, T is the set of agents and X_t the individual risk of agent $t \in T$. The process X is i.i.d., in the sense that $X_{t_1}, \dots, X_{t_n}$ are i.i.d. random variables for all $n \geqslant 1$ and all distinct $t_1, \dots, t_n \in T$. Also, T is viewed as "very large" and this is formalized by assuming Q nonatomic.

Let Γ denote the distribution common to the X_t. So, Γ is a Borel probability measure on $\mathbb{R}$ such that $X_t \sim \Gamma$ for all $t \in T$. Define also

$$\mathcal{H} = \{H \in \mathcal{C}\colon\ Q(H) > 0\}.$$

The informal idea underlying most economic models is that, for large T, individual risks disappear in the aggregate. To make this intuition precise, it is assumed that

$$X^\omega \text{ is measurable and } Q(X^\omega \in \cdot) = \Gamma(\cdot) \text{ for } P\text{-almost all } \omega \in \Omega. \tag{1}$$

Moreover, condition (1) is often strengthened as follows:

$$\text{for each } H \in \mathcal{H}, \text{ there is } A_H \in \mathcal{A} \text{ with } P(A_H) = 1 \text{ such that}$$
$$X^\omega \text{ is measurable and } Q(X^\omega \in \cdot \mid H) = \Gamma(\cdot) \text{ for } \omega \in A_H. \tag{2}$$

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In economics, condition (1) is usually called *law of large numbers* or else *no aggregate uncertainty*. Condition (2) was first emphasized by Feldman and Gilles in [4]. As each $H \in \mathcal{H}$ is a coalition of agents, following the suggestion of an anonymous referee, (2) may be called *coalitional aggregate certainty*. This note focus on (2).

It is not hard to prove that, when $\mathcal{C}$ is countably generated, condition (2) implies that Γ is 0–1 valued; see Section 1 of [4] and Theorem 4.2 of [7] (to make the paper self-contained, a proof is also given in Remark 2). Thus, to get (2) with nontrivial Γ and countably generated $\mathcal{C}$, an extension of $(T, \mathcal{C}, Q)$ is to be involved.

One (interesting) approach is to look for reasonable extensions, that is, extensions which grant (2) and some other properties, such as a form of Fubini's theorem. This route is followed by [6–8]. In Theorem 2.8 of [7], condition (2) is shown to be true if X is *essentially pairwise independent* and measurable with respect to a *Fubini extension* of the product σ-field. Conditions for such an X to exist are given in [6] and [8]. These conditions require $(T, \mathcal{C}, Q)$ to be extended if Γ is nontrivial and $\mathcal{C}$ countably generated.

A different route, closer to the ideas of [5], is taken in this note. On one hand, we aim to avoid extensions of $(T, \mathcal{C}, Q)$ and to obtain *any given* distribution for X. Thus, we do not require X i.i.d., but we allow $X \sim \mathcal{P}$ for any consistent set $\mathcal{P}$ of finite dimensional distributions (see Section 2 for precise definitions). On the other hand, we content ourselves with proving consistency of (2) with $X \sim \mathcal{P}$.

Our result is the following. As in most economic models, suppose $(T, \mathcal{C}, Q)$ is given with Q nonatomic and $\{t\} \in \mathcal{C}$ for all $t \in T$. In addition, fix a Borel probability measure Γ on $\mathbb{R}$ and a consistent set $\mathcal{P}$ of finite dimensional distributions. Note that Γ and $\mathcal{P}$ are now arbitrary and not necessarily connected.

Theorem 1. *If $(T, \mathcal{C}, Q)$, Γ and $\mathcal{P}$ are as above, there is a finitely additive probability space $(\Omega, \mathcal{A}, P)$ and a process $X : T \times \Omega \to \mathbb{R}$ such that $X \sim \mathcal{P}$ and condition* (2) *holds.*

In Theorem 1, Ω is the set of all functions $\omega : T \to \mathbb{R}$ and X the canonical process $X(t, \omega) = \omega(t)$. As first noted by Doob in [2], such an X is not measurable with respect to the product σ-field $\mathcal{C} \otimes \mathcal{G}$ where $\mathcal{G} = \sigma(X_t\colon\ t \in T)$. Other related results are Theorem 1 of [3], Proposition 3 of [4] and Propositions 6.1 and 6.4 of [7].

Dating from de Finetti, the finitely additive theory of probability is well founded and developed, even if not prevailing. It finds applications in various fields, ranging from statistics and number theory to economics. The spirit of Theorem 1 is that, in such theory, one can always assume condition (2) and $X \sim \mathcal{P}$ for any Γ and $\mathcal{P}$.

Plainly, as Γ and $\mathcal{P}$ are arbitrary, Theorem 1 may also lead to "absurd" claims. (Incidentally, this explains the title of this note.) If $T = [0, \infty)$, for instance, one could take $\Gamma = \delta_{x_0}$ for some $x_0 \in \mathbb{R}$ and $\mathcal{P}$ such that X is a Brownian motion.

However, in the subjective approach, the existence of different probability evaluations (modeling different opinions) should be viewed as a merit. It is a task of the economist to choose Γ and $\mathcal{P}$ in a reasonable way. Once the choice is done, in the economist's view, the question is: can I assume condition (2) and $X \sim \mathcal{P}$? In a finitely additive setting, the answer is: yes, you can, but any other choice of Γ and $\mathcal{P}$ (possibly meaningless or absurd) is consistent with (2) as well.

2. Proof and remarks

In this note, a collection $\mathcal{P}$ of finite dimensional distributions is meant as

$$\mathcal{P} = \big\{\mu(t_1, \ldots, t_n)\colon\ n \geqslant 1,\ t_1, \ldots, t_n \in T\big\}$$

where each $\mu(t_1, \ldots, t_n)$ is a Borel probability measure on $\mathbb{R}^n$. We write $X \sim \mathcal{P}$ if $X = \{X_t\colon\ t \in T\}$ is a real process, indexed by T, satisfying $(X_{t_1}, \ldots, X_{t_n}) \sim \mu(t_1, \ldots, t_n)$ for all $n \geqslant 1$ and $t_1, \ldots, t_n \in T$. We say that $\mathcal{P}$ is *consistent* in case $X \sim \mathcal{P}$ for some process X. Simple conditions for $\mathcal{P}$ to be consistent are given by the well-known Kolmogorov extension theorem.

An atom of Q is a set $C \in \mathcal{C}$ such that $Q(C) > 0$ and $Q(\cdot \mid C)$ is 0–1 valued. If Q has no atoms, it is called *nonatomic*. In case T is a separable metric space and $\mathcal{C}$ the Borel σ-field, Q is nonatomic if and only if $Q\{t\} = 0$ for all $t \in T$. Next, for each $H \subset T$, define the Q-outer measure $Q^*(H) = \inf\{Q(C)\colon\ H \subset C \in \mathcal{C}\}$. If $Q^*(H) = 1$, then Q can be extended to a probability measure Q_0 on $\sigma(\mathcal{C} \cup \{H\})$ such that $Q_0(H) = 1$.

We are now able to prove Theorem 1.

Proof of Theorem 1. Let Ω be the set of functions $\omega : T \to \mathbb{R}$ and X the canonical process $X(t, \omega) = \omega(t)$ for all $(t, \omega) \in T \times \Omega$. Let $\mathcal{G}$ be the σ-field on Ω generated by the maps $\omega \mapsto \omega(t)$ for all $t \in T$. Note that $X^\omega = \omega$ for all $\omega \in \Omega$. Also, since $\mathcal{P}$ is consistent, there is a probability measure $\mathbb{P}$ on $\mathcal{G}$ such that $X \sim \mathcal{P}$ under $\mathbb{P}$.

Let $\mathcal{H}_0 \subset \mathcal{H}$ be finite. Then, $\mathbb{P}$ can be extended to a probability measure P_0 such that

$$X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \bigm| H\big) = \Gamma(\cdot) \text{ for } H \in \mathcal{H}_0 \text{ and } P_0\text{-almost all } \omega. \tag{3}$$

The proof of (3) is similar to those of Theorem 2.2 of [2] and Proposition 6.1 of [7]. Define

$$A = \big\{\omega \in \Omega\colon\ X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \bigm| H\big) = \Gamma(\cdot) \text{ for all } H \in \mathcal{H}_0\big\}.$$

It suffices to prove $\mathbb{P}^*(A) = 1$. In turn, for $\mathbb{P}^*(A) = 1$, it suffices $A \neq \emptyset$ and