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A class of nonlinear degenerate elliptic equations related to the Gauss measure

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ABSTRACT

We prove *a priori* estimates and existence results for Dirichlet problems whose model case is:

$$\begin{cases} -\operatorname{div}\left(\dfrac{|Du|^{p-2}Du}{(1+|u|)^{\theta(p-1)}}\varphi\right) = g\varphi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open subset of $\mathbb{R}^N$ with $\gamma(\Omega) < 1$, $N \geqslant 2$, $1 < p < +\infty$, $\theta \geqslant 0$ and φ is the density of Gauss measure.

1. Introduction

In this paper we are interested in existence results for a class of nonlinear elliptic equations whose model case is

$$\begin{cases} -\operatorname{div}\left(\dfrac{|Du|^{p-2}Du}{(1+|u|)^{\theta(p-1)}}\varphi(x)\right) = g\varphi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^N$ with $\gamma(\Omega) < 1$, $N \geqslant 2$, $p > 1$, $\theta \geqslant 0$ and $\varphi(x) = (2\pi)^{-\frac{N}{2}} \exp(-\frac{|x|^2}{2})$ is the density of Gauss measure. Requiring different assumptions on the integrability of g we will obtain the existence of weak, distributional and entropy solutions. The boundary condition "$u = 0$" in weak sense means that u or its truncation belongs to $W_0^{1,q}(\Omega, \gamma)$ for some $q \geqslant 1$ (see Section 2.1 for definitions). Let us emphasize that the equation in (1.1) describes the stationary state of a diffusion process within a system due to random motions. Let us consider a random motion of a particle initially located in x at time t. Then the motion of the particle is described by its probability to exist in a specific volume element at time t. Typically $u\varphi$ has to be interpreted as the probability density function associated to the position of a single particle. If $p = 2$, V is a smooth region within Ω, ν is its outside normal, $J = \frac{Du}{(1+|u|)^{\theta}}$ is the flux density and g is a term that takes into account the possible existence of a source in V, then equation in (1.1) is equivalent to

$$\int_{\partial V} J \cdot \nu\varphi \, d\sigma = \int_V g\varphi \, dx.$$

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We note that the diffusion coefficient depends on the function $\frac{1}{(1+|u|)^\theta}$, hence it goes to zero, *i.e.* a slow diffusion effect appears, for large value of u (see *e.g.* [34,37,16] and the references therein). If $p=2$ and $\theta=0$, the equation in (1.1) describes the Ornstein–Uhlenbeck process. It is well known that this process is a model of Brownian motion and the interest in this kind of model arises in many applications (for instance in financial mathematics and for modelling biological processes, see *e.g.* [29]).

Because of the presence of density φ we have to work in weighted Sobolev spaces and because of the presence of the function $b(|u|)=\frac{1}{(1+|u|)^{\theta(p-1)}}$ the operator $A(u)=-\operatorname{div}(\frac{|Du|^{p-2}Du}{(1+|u|)^{\theta(p-1)}}\varphi)$ may not be coercive on $W_0^{1,p}(\Omega,\gamma)$ (see Section 2 for definition).

To overcome this problem we reason by approximation, truncating the function b in order to obtain operators in the class of Leray–Lions. We get suitable *a priori* estimates for the solutions of these approximated problems using symmetrization methods (see *e.g.* [35]). Moreover, due to the presence of the function φ and the possible unboundedness of the domain Ω, we use the isoperimetric inequality and the notion of rearrangement with respect to the Gauss measure $d\gamma=\varphi(x)\,dx$.

Thanks to a combined use of *a priori* estimates and the almost everywhere convergence of gradient of solutions of approximated problems, it is possible to pass to the limit in order to obtain existence results.

When Ω is bounded and $\varphi\equiv 1$ *a priori* estimates and existence results have been obtained by several authors using different techniques (see for example [2,3,10,12,13,32]). More precisely, if $0\leqslant\theta<1$ it has been proved in [2], that if g belongs to $L^m(\Omega)$ with $m\geqslant\frac{Np}{(N-p)(1-\theta)(p-1)+p^2}$, then there exists a solution in the energy space $W_0^{1,m}(\Omega)$, and such solution is also bounded if $m>N/p$. If the datum g is less regular, then it is proved the existence of distributional or entropy solutions (see also [13,12,20] for the linear case $p=2$). We stress that these results coincide with the classical one when there is no degeneracy, that means $\theta=0$. Completely different is the case $\theta>1$, in which the existence of weak bounded solutions is related to a smallness assumption on the L^m-norm of g, with $m>N/p$ (see [2]). Finally in the case $\theta=1$, it is possible to prove the existence of a bounded solution if g belongs to L^m, with $m>N/p$ (see, for example, [2,3,12]). The parabolic case has been studied in [19] and [34].

When $\theta=0$ in problem (1.1) the weighted Zygmund spaces (see Section 2 for the definitions) are the natural sets for the datum to get existence results. More precisely, in [23] has been proved that if the datum g is in the Zygmund space $L^{\frac{p}{p-1}}(\log L)^{-\frac{1}{2}}(\Omega,\gamma)$ then there exists a weak solution in the energy space $W_0^{1,p}(\Omega,\gamma)$. When the datum is less regular, it has been proved that if g belongs to $L^{\frac{m}{p-1}}(\log L)^{-\frac{1}{2}}(\Omega,\gamma)$, with $p-1<m<p$, then there exists a weak solution in the sense of distribution in $W_0^{1,m}(\Omega,\gamma)$ (see [24]). Moreover if g is in $L^1(\log L)^{\frac{1}{2}}(\Omega,\gamma)$, then there exists a unique SOLA (Solution Obtained as Limit of Approximations) in $W_0^{1,p-1}(\Omega,\gamma)$ (see [25]). The parabolic case has been studied in [17].

In the present paper we examine the question of existence in the case $\theta\neq 0$ varying the summability of b and taking g in the class of Zygmund spaces. More precisely in the case $0<\theta<1$, we prove the existence of weak solutions in the energy space $W_0^{1,p}(\Omega,\gamma)$ when the datum g belongs to $L^{\frac{m}{p-1}}(\log L)^{\alpha(p-1)-\frac{p}{2}}(\Omega,\gamma)$, with $m\geqslant\frac{p}{1-\theta}$ and suitable α. Moreover such solution is bounded if g is in the Lorentz–Zygmund space $L^{\infty,\frac{1}{p-1}}(\log L)^{-\frac{p}{2}}(\Omega,\gamma)$ with $1<p<2$. If g is less regular we prove the existence of distributional and entropy solutions to problem (1.1). On the other hand, if $\theta>1$ in order to prove the existence of weak bounded solutions in $W_0^{1,p}(\Omega,\gamma)$ we need a smallness assumption on the datum.

The paper is organized as follows. In Section 2 we give the assumptions on the data of the problem and we adapt different definitions of solution to the context of the Gauss measure. Moreover we recall tools that are useful in what follows. In Sections 3 and 4 we prove the differential inequality for solutions to suitable approximated problems in terms of their decreasing rearrangements. Starting from this inequality, we prove *a priori* estimates for the gradient of solution of approximated problem. Finally in Section 5 we prove the existence results quoted above.

2. Preliminaries

2.1. Statement of the problem and definitions of solutions

We deal with the following nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}\big(a(x,u,Du)\big)=g\varphi & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is an open subset of $\mathbb{R}^N$, $N\geqslant 2$ with $\gamma(\Omega)<1$, $1<p<\infty$ and

$$\varphi(x)=(2\pi)^{-\frac{N}{2}}\exp\left(-\frac{|x|^2}{2}\right)$$

is the density of Gauss measure $\mathbb{R}^N$ defined as $d\gamma=\varphi(x)\,dx$.