J. Math. Anal. Appl. 382 (2011) 534–548

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On boundedness of discrete multilinear singular integral operators

Paco Villarroya [1]

School of Mathematics, University of Edinburgh, King's Buildings, EH10 3JZ, Edinburgh, UK

ARTICLE INFO

Article history:
Received 25 November 2009
Available online 15 January 2011
Submitted by Richard M. Aron

Keywords:
Multipliers
Transference
Lorentz spaces
Bilinear Hilbert transforms

ABSTRACT

Let $m(\xi,\eta)$ be a measurable locally bounded function defined in $\mathbb{R}^2$. Let $1 \leqslant p_1, q_1, p_2, q_2 < \infty$ such that $p_i = 1$ implies $q_i = \infty$. Let also $0 < p_3, q_3 < \infty$ and $1/p = 1/p_1 + 1/p_2 - 1/p_3$. We prove the following transference result: the operator

$$\mathcal{C}_m(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i x(\xi+\eta)}\,d\xi\,d\eta$$

initially defined for integrable functions with compact Fourier support, extends to a bounded bilinear operator from $L^{p_1,q_1}(\mathbb{R}) \times L^{p_2,q_2}(\mathbb{R})$ into $L^{p_3,q_3}(\mathbb{R})$ if and only if the family of operators

$$\mathcal{D}_{\widetilde{m}_{t,p}}(a,b)(n) = t^{\frac{1}{p}} \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} P(\xi)Q(\eta)m(t\xi,t\eta)e^{2\pi i n(\xi+\eta)}\,d\xi\,d\eta$$

initially defined for finite sequences $a = (a_{k_1})_{k_1\in\mathbb{Z}}$, $b = (b_{k_2})_{k_2\in\mathbb{Z}}$, where $P(\xi) = \sum_{k_1\in\mathbb{Z}} a_{k_1}e^{-2\pi i k_1\xi}$ and $Q(\eta) = \sum_{k_2\in\mathbb{Z}} b_{k_2}e^{-2\pi i k_2\eta}$, extend to bounded bilinear operators from $l^{p_1,q_1}(\mathbb{Z}) \times l^{p_2,q_2}(\mathbb{Z})$ into $l^{p_3,q_3}(\mathbb{Z})$ with norm bounded by uniform constant for all $t > 0$. We apply this result to prove boundedness of the discrete Bilinear Hilbert transforms and other related discrete multilinear singular integrals including the endpoints.

1. Introduction

Linear multiplier operators can be defined over a large variety of groups in the following way: given G a locally compact abelian group with Haar measure μ and dual group $\hat{G}$, a measurable function m that takes values in $\hat{G}$ defines a multiplier operator if for every $f \in L^p(G)$ there exists $g \in L^p(G)$ such that $\mathcal{F}g = m \cdot \mathcal{F}f$. Here $\mathcal{F}$ is the continuous extension to $L^p(G)$ of the Fourier transform operator, initially defined in $L^1(G) \cap L^p(G)$ as $\mathcal{F}f(\gamma) = \int_G f(x)\gamma(-x)\,d\mu(x)$ for every $\gamma \in \hat{G}$. The multiplier operator is then defined by $T_m(f) = g$.

This way, it is a very natural question to ask about the possibility of transferring the boundedness properties of such operators when they are defined over two different groups. That is, given a multiplier operator T_m known to be bounded between spaces defined over certain groups, let's say from $L^p(G_1)$ to $L^q(G_2)$, we want to know when the analogous operator T'_m defined over different groups is also bounded between similar type of spaces, let's say from $L^p(G'_1)$ to $L^q(G'_2)$.

E-mail address: paco.villarroya@ed.ac.uk.

[1] The author has been partially supported by grants MTM 2005-08350-C03-03 and MTM2008-04594.

0022-247X/$ – see front matter © 2011 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2011.01.025

The first transference methods for linear multipliers were given by K. de Leeuw [24] who showed that if m is a bounded measurable function which is pointwise limit of continuous functions then the linear operator

$$T_m(f)(x) = \int_{\mathbb{R}} \hat{f}(\xi)m(\xi)e^{2\pi i x\xi}\,d\xi$$

defined for $f \in S(\mathbb{R})$, extends boundedly to $L^p(\mathbb{R})$ with $1 \leqslant p < \infty$ if and only if

$$\tilde{T}_{m_\varepsilon}(f)(\theta) = \sum_{k\in\mathbb{Z}} \hat{f}(k)m(\varepsilon k)e^{2\pi i k\theta}$$

defined for periodic functions f, extend to uniformly bounded operators on $L^p(\mathbb{T})$ for all $\varepsilon > 0$.

Other types of linear transference theorems were given by P. Auscher and M.J. Carro (see [1]) on Lebesgue spaces between $\mathbb{R}^n$ and $\mathbb{Z}^n$. They proved that if m is a measurable bounded function then the operator

$$T_m(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)m(\xi)e^{2\pi i x\xi}\,d\xi$$

defined for $f \in S(\mathbb{R}^n)$, extends boundedly to $L^p(\mathbb{R}^n)$ with $1 \leqslant p < \infty$ if and only if

$$\bar{T}_{m_\varepsilon}(a)(k) = \int_{[-\frac{1}{2},\frac{1}{2}]^n} P(\xi)m(\varepsilon\xi)e^{2\pi i k\xi}\,d\xi$$

for $a = (a_k)_{k\in\mathbb{Z}}$ and $P(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k\xi}$, extend to uniformly bounded operators on $l^p(\mathbb{Z}^n)$ for all $\varepsilon > 0$.

The interest for multilinear multipliers, which in the case of the real line can be defined as

$$C_m(f_1, f_2, \ldots, f_n)(x) = \int_{\mathbb{R}^n} \hat{f}_1(\xi_1)\cdots\hat{f}_n(\xi_n)m(\xi_1, \xi_2, \ldots, \xi_n)e^{2\pi i x(\xi_1+\xi_2+\cdots+\xi_n)}\,d\xi$$

for $f_i \in \mathcal{S}(\mathbb{R})$, started in the seventies with the works of R. Coifman and Y. Meyer. They proved (see [11–13]) boundedness for multilinear multipliers whose symbols m have singularities at most at a single point. At the end of the nineties, M. Lacey and C. Thiele [21,22] proved that the bilinear Hilbert transforms, a family of bilinear multipliers for which $m(\xi, \nu) = \operatorname{sign}(\xi + \alpha\nu)$, are bounded multipliers from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$ for $1 < p_1, p_2 \leqslant \infty$, $p_3 > 2/3$ and $\alpha \in \mathbb{R}\backslash\{0, 1\}$. Their paper was the first one with a proof of boundedness for multilinear multipliers whose symbols have singularities spread over large sets. This seminal work was quickly followed by many different extensions and generalizations. See the works by Grafakos and Li [18] and Li [25], by J.E. Gilbert and A.R. Nahmod [16,17], by C. Muscalu, T. Tao and C. Thiele [26,27], by M. Lacey [23] and by Grafakos, Tao and Terwilleger [19].

Multilinear multiplier operators can also be defined over different groups and so the question of transference of boundedness properties also applies to them. This way, D. Fan and S. Sato (see [15]) proved the multilinear version of the transference between $\mathbb{R}$ and $\mathbb{Z}$, namely that continuous functions $m(\xi, \eta)$ define multiplier operators of strong and weak type (p_1, p_2) on $\mathbb{R} \times \mathbb{R}$ if and only if $(m(\varepsilon k, \varepsilon k'))_{k,k'\in\mathbb{Z}}$ define a uniformly bounded family of multipliers of strong and weak type (p_1, p_2) on $\mathbb{Z} \times \mathbb{Z}$.

Other references addressing the issue of transference of linear or multilinear multiplier operators through several different methods are the following papers [5,7,9,10] and also the classic text [14].

The aim of the present paper is to get an extension of Auscher and Carro's result in the multilinear setting for multipliers acting on Lorentz spaces which, in some sense, completes the Lorentz transferences proven in [6] between $\mathbb{R}$ and $\mathbb{T}$. Unlike the linear case, in the multilinear setting many interesting operators are known to be bounded in Lebesgue spaces with exponents below one and so these cases need to be included in the transference results. This feature and the fact of dealing with Lorentz norms are the main difficulties and novelties in the present work. Although all results hold true for multilinear multipliers in spaces of several variables $\mathbb{R}^n$ and $\mathbb{Z}^n$, for the sake of simplicity in notation we restrict ourselves to bilinear operators with argument functions of one real variable.

We apply the transference results to prove $l^{p_1}(\mathbb{Z}) \times l^{p_2}(\mathbb{Z})$ into $l^{p_3}(\mathbb{Z})$ boundedness of the discrete Bilinear Hilbert transforms, defined for any two finite sequences a, b as

$$\mathcal{H}_\alpha(a, b)(n) = \frac{1}{\pi}\sum_{k\in\mathbb{Z},\ k\neq 0} a_{n-k}b_{n-\alpha k}\frac{1}{k}$$

with $\alpha \in \mathbb{Z}\backslash\{0, 1\}$. This result has been previously proven by other methods (see [7] and specially [5]) when $\alpha = -1$. We now manage to transfer boundedness for the whole family of operators with $p_3 > 2/3$. Moreover, for other exponents the transference result also holds although their boundedness properties have not been established yet.