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# Riesz means associated with certain product type convex domain ☆

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ABSTRACT

In this paper we study the maximal operators $T_*^\delta$ and the convolution operators $T^\delta$ associated with multipliers of the form

$$\big(1-\max\big\{|\xi_0|,|\xi_1|,\ldots,|\xi_{n-2}|\big\}\big)_+^\delta,\quad (\xi_0,\xi_1,\ldots,\xi_{n-2})\in\mathbb{R}^2\times\mathbb{R}^{n-2},\ n\geqslant 3.$$

We prove that $T_*^\delta$ satisfies the sharp weak type $(p,p)$ inequality on $H^p(\mathbb{R}^n)$ when $\frac{2(n-1)}{2n-1}<p<1$ and $\delta=n(\frac{1}{p}-1)+\frac{1}{2}$, or when $p=\frac{2(n-1)}{2n-1}$ and $\delta>n(\frac{1}{p}-1)+\frac{1}{2}$. We also obtain that $T^\delta$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $\delta>\max\{2|\frac{1}{p}-\frac{1}{2}|-\frac{1}{2},0\}$ and $1\leqslant p<\infty$. The indicated ranges of parameters $p$ and $\delta$ cannot be improved.

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## 1. Introduction

For a Schwartz function $f\in\mathcal{S}(\mathbb{R}^n)$ we denote by $\widehat{f}$ the Fourier transform of $f$. For $n\geqslant 3$ we define a distance function $\varrho$ as

$$\varrho(\xi)=\max\big\{|\xi_0|,|\xi_1|,\ldots,|\xi_{n-2}|\big\},\quad \xi=(\xi_0,\xi_1,\ldots,\xi_{n-2})\in\mathbb{R}^2\times\mathbb{R}^{n-2}.$$

For $\delta>0$ we consider a convolution operators $T_\epsilon^\delta$ by

$$\widehat{T_\epsilon^\delta f}(\xi)=\left(1-\frac{\varrho(\xi)}{\epsilon}\right)_+^\delta\widehat{f}(\xi)\tag{1.1}$$

and the maximal operators $T_*^\delta$ by

$$T_*^\delta f(x)=\sup_{\epsilon>0}\big|T_\epsilon^\delta f(x)\big|,\quad x=(x_0,x_1,\ldots,x_{n-2})\in\mathbb{R}^2\times\mathbb{R}^{n-2},\ n\geqslant 3.$$

Here $t_+^\delta=t^\delta$ for $t>0$ and zero otherwise. In this paper we consider the multiplier $m$ defined by

$$m(\xi)=\big(1-\max\big\{|\xi_0|,|\xi_1|,\ldots,|\xi_{n-2}|\big\}\big)_+^\delta,$$

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which is supported in the product type convex domain $D \times I \times \cdots \times I$, where $D$ is the unit disc in $\mathbb{R}^2$ centered at the origin and $I = [-1, 1]$. In this paper we are aiming to establish sharp weak type estimates for the maximal operator $T_*^\delta$ and strong type estimates for the convolution operators $T^\delta$ associated with a multiplier $m$. One of the important parts of doing this is to analyze the kernel function which is the inverse Fourier transform of the multiplier $m$. Especially when $p < 1$ decay estimates of the kernel immediately give us the desired estimates if we combine them with standard arguments developed by Stein, Taibleson and Weiss in [16]. The decay of the kernel is related with the differentiability of the multiplier so delicate analysis near points where the differentiability of the multiplier breaks down is necessary for what we desire to obtain. Even though estimates for the complementary case carry a different story, understanding the singular set of the multiplier enables us to successfully perform appropriate dyadic decompositions of the kernel. One can put together optimal estimates for each dyadic piece to obtain results for the original kernel without any loss. Hence it is worth noticing that the differentiability of the multiplier breaks down on the singular set

$$\mathcal{R} = \{\xi\colon |\xi_i| = |\xi_j|\} \cup \{\xi\colon \varrho(\xi) = 1\}. \tag{1.2}$$

As we see that $\mathcal{R}$ is a union of two sets. Intuitively, the differentiability of the multiplier on the set $\{\xi\colon \varrho(\xi) = 1\}$ can be improved when we take bigger $\delta$. However the size of $\delta$ does not affect the differentiability of the multiplier on the complementary set $\{\xi\colon |\xi_i| = |\xi_j|\}$. These roughly indicate that the singular set $\{\xi\colon \varrho(\xi) = 1\}$ will give the restriction of $\delta$ and the set $\{\xi\colon |\xi_i| = |\xi_j|\}$ will give the other restriction, which turns out to be the restriction on the range of $p$.

One more useful observation to comprehend the multiplier is as follows:

The distance function $\varrho(\xi)$ is equal to lower dimensional Euclidean distance function $|\xi_\ell|$ in conical regions $\varrho(\xi) = |\xi_\ell|$. So in each conical region the multiplier behaves like lower dimensional Bochner–Riesz multiplier. For example, we consider the case $n = 3$. In this case $\varrho(\xi) = \max\{|\xi_0|, |\xi_1|\}$ where $\xi = (\xi_0, \xi_1) \in \mathbb{R}^2 \times \mathbb{R}$ and the multiplier is equal to $\omega(\xi) = (1 - \max\{|\xi_0|, |\xi_1|\})_+^\delta$. If $|\xi_0| > |\xi_1|$, $\omega(\xi) = (1 - |\xi_0|)_+^\delta$ which is the multiplier for 2-dimensional spherical Bochner–Riesz means. If $|\xi_0| < |\xi_1|$, $\omega(\xi) = (1 - |\xi_1|)_+^\delta$ which is the multiplier for one-dimensional Bochner–Riesz means. When $|\xi_0| = |\xi_1|$, the multiplier $\omega$ is not smooth on the cone $\{(\xi_0, \xi_1)\colon |\xi_0| = |\xi_1|\}$. So the operator is a combination of 1 and 2-dimensional Bochner Riesz means and the cone multipliers in three-dimensional Euclidean space.

H. Luers has investigated this type of operator in a different setting in [8]. He obtained $L^p$ estimates of convolution operators $S^\delta$ associated with a cylinder multiplier defined by

$$\widehat{S^\delta f}(\xi) = \left(1 - \max\{|\xi_0|, |\xi_1|\}\right)_+^\delta \widehat{f}(\xi), \quad \xi = (\xi_0, \xi_1) \in \mathbb{R}^k \times \mathbb{R},$$

where $k \geqslant 2$. We note that when $k \geqslant 3$ the multiplier is related with higher dimensional Bochner–Riesz multiplier. His results are not sharp and it is conjectured that $S^\delta$ is bounded for all $\delta > \delta(p) := \max\{k|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$ and $\delta(p) < 1$. P. Taylor recently improved the Luers' result in [17] by adapting the arguments in [1,9,10,18].

As for $p < 1$, it has been obtained for $k = 2$ that the maximal cylinder operator is of weak type $(p, p)$ on $H^p(\mathbb{R}^3)$ when $\frac{4}{5} < p < 1$ and $\delta = 3(\frac{1}{p} - 1) + \frac{1}{2}$, or when $p = \frac{4}{5}$ and $\delta > 3(\frac{1}{p} - 1) + \frac{1}{2}$ in [7]. It has been also shown that the above estimates are sharp. For higher dimensional cases $k \geqslant 3$, the range of $p$ for the boundedness of $S^\delta$ does not intersect with $p < 1$. In these case the kernels are not even integrable for any admissible $\delta$ (see [8]). Hence one cannot expect any result in the range $p < 1$ when $k \geqslant 3$. Even for $p \geqslant 1$, any estimate for higher dimensional Bochner–Riesz means and cone multipliers has not been sharp.

One of the main purposes of this paper is to investigate the relations between the geometry of singular sets and restrictions on the range of both $p$ and $\delta$ via careful analysis. We are interested in taking a look at both cases: $p \geqslant 1$ and $p < 1$. These have led us to take account of the case $k = 2$, that is, the case where the multiplier is associated with a product of a disc and intervals.

When the operator is defined by the multiplier associated with a product of intervals, similar restrictions on $p$ and $\delta$ have been observed by P. Oswald in [12,13]. However the existence of a disc in the product causes a more interesting feature of the operator due to the non-vanishing curvature of the boundary of the disc. The results in [7,17] are on the multiplier associated with a product of a disc and one interval and those in [12,13] are on the multipliers associated with a product of only intervals. Even though operators of the first type conveys property caused by the curvature, operators of the second type also have interesting mapping property because of the interactions between intervals. It would be interesting to see all phenomena at once.

In terms of method we develop ideas in [7,17]. However our situation is much more complicated and general than that in [7,17]. We need more systematic approach to obtain the kernel estimates. Once we obtain optimal kernel estimates, the integrability of the kernel is immediate. This helps us to obtain $L^p$ estimates when $p \geqslant 1$. If we just follow the idea in [7], then we might be in trouble because there are too many cases which we have to deal with and this will make things to be complicated. To conquer this difficulty we have to figure out essential role of each procedure made to obtain kernel estimates in [7]. As soon as one completely analyze arguments for the kernel estimates in [7], one can simplify the argument enough to deal with much more general case. This is one of the points which distinguishes our results from those in [7,17].

Now we describe how the arguments will be processed to realize all mentioned before. In view of the dilation property of the kernel of $T_\epsilon^\delta$ it suffices to consider the kernel of $T_1^\delta$. For the sake of notational convenience we set $T^\delta = T_1^\delta$ in (1.1). To obtain the sharp decay of the kernel we decompose the multiplier. Firstly, we decompose the multiplier into

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