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Existence of positive periodic solutions of nonlinear first-order delayed differential equations

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ABSTRACT

We consider the existence of positive ω-periodic solutions for the equation

$$u'(t) = a(t)g\big(u(t)\big)u(t) - \lambda b(t)f\big(u\big(t-\tau(t)\big)\big),$$

where $a, b \in C(\mathbb{R}, [0,\infty))$ are ω-periodic functions with $\int_0^\omega a(t)\,dt > 0$, $\int_0^\omega b(t)\,dt > 0$; $f, g \in C([0,\infty), [0,\infty))$ and $f(s) > 0$ for $s > 0$; τ is a continuous ω-periodic function; $\lambda > 0$ is a parameter. The proofs of our main results are based upon fixed point index theory.

1. Introduction

In recent years, there has been considerable interest in the existence of periodic solutions of the following equation

$$u'(t) = a(t)g\big(u(t)\big)u(t) - \lambda b(t)f\big(u\big(t-\tau(t)\big)\big), \tag{1.1}$$

where $a, b \in C(\mathbb{R}, [0,\infty))$ are ω-periodic functions, $\int_0^\omega a(t)\,dt > 0$, $\int_0^\omega b(t)\,dt > 0$, τ is a continuous ω-periodic function. (1.1) has been proposed as a model for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias. See, for example, [1–13] and the references therein.

The existence results in the literature are largely based on the assumption that there exist two positive constants l and L, such that

$$0 < l \leqslant g(u) \leqslant L. \tag{1.2}$$

It is interesting to know whether there is a positive solution of (1.1) when g does not satisfy (1.2). Very recently, Jin and Wang [13] studied the existence of positive ω-periodic solutions of the spectral equation

$$u'(t) = a(t)e^{u(t)}u(t) - \lambda b(t)f\big(u\big(t-\tau(t)\big)\big) \tag{1.3}$$

under the assumptions:

(H1) $a, b \in C(\mathbb{R}, [0,\infty))$ are ω-periodic functions, $\int_0^\omega a(t)\,dt > 0$, $\int_0^\omega b(t)\,dt > 0$, τ is a continuous ω-periodic function, $g : [0,\infty) \to [0,\infty)$ is continuous;
(H2) $f \in C([0,\infty), [0,\infty))$ and $f(s) > 0$ for $s > 0$;

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(H3) $\sigma := e^{-\int_0^\omega a(t)\,dt} < 1$, and for $r > 0$,

$$m(r) := \min\left\{ f(u)\colon\ \frac{\sigma^{e^r}(1-\sigma)}{1-\sigma^{e^r}} r \leqslant u \leqslant r \right\} > 0.$$

They proved the following

Theorem A. *Assume that* (H1)–(H3) *hold and* $\lim_{u\to 0^+} \frac{f(u)}{u} = 0$. *Then for each* $L > 0$, *there exists a* $\lambda_0 > 0$ *such that* (1.3) *has a positive* ω*-periodic solution* u *with* $\sup_{t\in[0,\omega]} u(t) \leqslant L$ *for* $\lambda > \lambda_0$.

It is easy to see that the function $g(u) = e^u$ satisfies

$$1 \leqslant e^u < \infty. \tag{1.4}$$

Of course, natural questions are

Q1. Whether or not a similar result can be proved under the more general condition

$$0 < l \leqslant g(u) < \infty? \tag{1.5}$$

Q2. Whether or not a similar result can be proved under the more general condition

$$0 < g(u) \leqslant L? \tag{1.6}$$

Q3. Under the assumptions (1.5) or (1.6), what will happen if we replace the assumption "$f_0 = 0$" with one of the assumptions: $f_0 = \infty$, $f_\infty = 0$, $f_\infty = \infty$, where

$$f_0 = \lim_{u\to 0} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}?$$

In this paper, we will give positive answers to Q1 and Q2, and give a partial answer to Q3. More precisely, in Sections 2–3, we will prove the existence of positive ω-periodic solutions for (1.1) under (1.5) and f satisfying one of the following conditions:

$$f_0 = 0, \qquad f_0 = \infty. \tag{1.7}$$

Sections 4–5 are devoted to studying the existence of positive ω-periodic solutions for (1.1) under (1.6) and one of the following conditions:

$$f_0 = 0, \qquad f_0 = \infty. \tag{1.8}$$

However, we give no any information on the existence of positive ω-periodic solutions for (1.1) in the four following cases:

(i) (1.5) holds and $f_\infty = \infty$;
(ii) (1.5) holds and $f_\infty = 0$;
(iii) (1.6) holds and $f_\infty = 0$;
(iv) (1.6) holds and $f_\infty = \infty$.

The following well-known result of the fixed point index is crucial in our arguments.

Theorem B. *(See [14–16].) Let* E *be a Banach space and* K *a cone in* E. *For* $r > 0$, *define* $K_r = \{u \in K\colon\ \|u\| < r\}$. *Assume that* $T : \bar{K}_r \to K$ *is completely continuous such that* $Tu \neq u$ *for* $u \in \partial K_r = \{u \in K\colon\ \|u\| = r\}$.

(i) *If* $\|Tu\| > \|u\|$ *for* $u \in \partial K_r$, *then*

$$i(T, \bar{K}_r, K) = 0.$$

(ii) *If* $\|Tu\| < \|u\|$ *for* $u \in \partial K_r$, *then*

$$i(T, \bar{K}_r, K) = 1.$$

Finally, let us denote that for each fixed constant $\rho > 0$,

$$h^*(\rho) := \max\{ g(s) \mid 0 \leqslant s \leqslant \rho \}, \tag{1.9}$$

$$h_*(\rho) := \min\{ g(s) \mid 0 \leqslant s \leqslant \rho \}. \tag{1.10}$$