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# Existence of solutions for quasilinear elliptic exterior problem with the concave-convex nonlinearities and the nonlinear boundary conditions

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ABSTRACT

In this paper, we consider the following quasilinear elliptic exterior problem

$$\begin{cases} -div\big(a(x)|\nabla u|^{p-2}\nabla u\big)+g(x)|u|^{q-2}u=h(x)|u|^{s-2}u+\lambda H(x)|u|^{r-2}u, & x\in\Omega,\\ a(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial\nu}+b(x)|u|^{p-2}u=0, & x\in\Gamma=\partial\Omega \end{cases}$$

where $\Omega$ is a smooth exterior domain in $\mathbb{R}^N$, and $\nu$ is the unit vector of the outward normal on the boundary $\Gamma$, $1<p<N$, $1<s<p<r<p^*=Np/(N-p)$. By the variational principle and the Mountain Pass Theorem, we establish the existence of infinitely many solutions if $q>r$ and at least one solution if $1<q<s$.



## 1. Introduction and the main results

Recently, Filippucci et al. in [8] consider the existence and nonexistence of solutions to the elliptic exterior problem

$$\begin{cases} -div\big(a(x)|\nabla u|^{p-2}\nabla u\big)+|u|^{q-2}u=\lambda H(x)|u|^{r-2}u, & x\in\Omega,\\ a(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial\nu}+b(x)|u|^{p-2}u=0, & x\in\Gamma=\partial\Omega \end{cases}\tag{1.1}$$

where $\Omega$ is a smooth exterior domain in $\mathbb{R}^N$ $(N\geqslant 3)$, and $\nu$ is the unit vector of the outward normal on the boundary $\Gamma=\partial\Omega$, $\lambda\geqslant 0$, $1<p<N$. The functions $a(x)\geqslant a_0>0$ and $H(x)$, $b(x)$ verify that

$(H1)$ $H(x)\in L^{p_0}(\Omega)\cap L^\infty(\Omega)$ is a nonnegative function which is positive on a non-empty open subset of $\Omega$, where $p_0=p^*/(p^*-r)$, $p^*=NP/(N-p)$.
$(H2)$ $b(x)$ is a continuous positive function on $\Gamma$.

By the variational method, they obtained the following main results:

(1) Let $p<r<q<p^*$. Then there exists $\lambda^*>0$ such that problem (1.1) has no nontrivial weak solution if $\lambda\leqslant\lambda^*$; and problem (1.1) has at least a nontrivial positive weak solution $u$ if $\lambda\geqslant\lambda^*$.
(2) Let $p<q<r<p^*$. Then problem (1.1) has no nontrivial weak solution if $\lambda\leqslant 0$ and has at least a nontrivial weak solution if $\lambda>0$.

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We note that the nonlinear terms in (1.1) are $|u|^{q-2}u$ and $\lambda H|u|^{r-2}u$ with $q > p$, $r > p$. But, in many cases, the nonlinearity term is the concave-convex function and it has received a great deal of attention in the recent years, see [12,13,16,17] and references therein.

In this paper, we study the following quasilinear elliptic exterior problem:

$$\begin{cases} -div\big(a(x)|\nabla u|^{p-2}\nabla u\big) + g(x)|u|^{q-2}u = h(x)|u|^{s-2}u + \lambda H(x)|u|^{r-2}u, & x \in \Omega, \\ a(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + b(x)|u|^{p-2}u = 0, & x \in \Gamma = \partial\Omega \end{cases} \tag{1.2}$$

where $\Omega$ is a smooth exterior domain in $\mathbb{R}^N$, $1 < p < N$, $1 < s < p < r < p^* = Np/(N-p)$. The function $a(x) \in C^{0,\delta}(\bar{\Omega}) \cap L^\infty(\bar{\Omega})$ is a positive function, $b(x) \in C(\Gamma) \cap L^\infty(\Gamma)$ is positive, $g(x) > 0$ in $\Omega$. Obviously, the nonlinearities in (1.2) contain concave and convex functions.

Elliptic equations involving the $p$-Laplacian operator arise in some physical models like the flow of non-Newtonian fluids: pseudo-plastic fluids correspond to $p \in (1,2)$ while dilatant fluids correspond to $p > 2$. The case $p = 2$ expresses Newtonian fluids, see [3]. On the other hand, quasilinear elliptic problems like (1.2) appear naturally in several branches of pure and applied mathematics, such as the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [7,15]); non-Newtonian fluids, reaction diffusion problems, flow through porous media, nonlinear elasticity, glaciology, etc. (see [3,6]). Also, problem (1.2) may be viewed as a prototype of pattern formation in biology and is related to the steady-state problem for a chemotactic aggregation model introduced by Keller and Segel in [11].

The motivation for our investigation is to consider the effect on the existence of solution if the nonlinear term $\lambda H(x)|u|^{r-2}u$ in (1.1) is replaced by $h(x)|u|^{s-2}u + \lambda H(x)|u|^{r-2}u$. Similar consideration can be found in [1,4,5,9,18]. We will find that problem (1.2) admits infinitely many solutions for every $\lambda \in \mathbb{R}^1$ if $r < q$. This is somewhat surprising, as compared with Theorem 1.1 in [8].

In this paper, we are concerned with the existence of nontrivial solutions for problem (1.2). Two cases will be considered. Precisely speaking, we will establish the existence of infinitely many solutions for $q > r$, and at least a nontrivial solution for $q < s$. We will use the variational method to study problem (1.2).

As in [8], we let $E$ be the completion of the restriction on $\Omega$ of functions of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{a,b} = \left(\int_\Omega a(x)|\nabla u|^p\,dx + \int_\Gamma b(x)|u|^p\,dS\right)^{1/p}. \tag{1.3}$$

The natural functional space to study problem (1.2) is $X = E \cap L^q(\Omega, g)$ with respect to the norm

$$\|u\| = \|u\|_X = \big(\|u\|_{a,b}^p + \|u\|_{L^q(\Omega,g)}^p\big)^{1/p} \tag{1.4}$$

where and in the sequel,

$$L^t(\Omega,\rho) = \left\{u(x)\ \Big|\ \int_\Omega \rho(x)\big|u(x)\big|^t\,dx < \infty\right\}, \qquad \|u\|_{L^t(\Omega,\rho)} = \left(\int_\Omega \rho(x)\big|u(x)\big|^t\,dx\right)^{1/t} \tag{1.5}$$

with $\rho(x) > 0$ in $\Omega$ and $t \geqslant 1$.

Then $X$ is the reflexive Banach space endowed with the norm $\|u\|$. We will see that the space $X$ is compactly embedded into the weighted Lebesgue space $L^s(\Omega,h)$ and $L^r(\Omega,H)$ by Lemma 2 below.

**Definition 1.** A function $u \in X$ is said to be a weak solution of problem (1.2) if

$$\begin{aligned} &\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \int_\Gamma b(x)|u|^{p-2}u\varphi\,dS + \int_\Omega g(x)|u|^{q-2}u\varphi\,dx \\ &\quad = \int_\Omega h(x)|u|^{s-2}u\varphi\,dx + \lambda\int_\Omega H(x)|u|^{r-2}u\varphi\,dx, \quad \forall\varphi \in X. \end{aligned} \tag{1.6}$$

It is clear that problem (1.2) has a variational structure. Let $J(u): X \to \mathbb{R}^1$ be the corresponding energy functional of problem (1.2) which is defined by

$$J(u) = \frac{1}{p}\|u\|_{a,b}^p + \frac{1}{q}\int_\Omega g(x)|u|^q\,dx - \frac{1}{s}\int_\Omega h(x)|u|^s\,dx - \frac{\lambda}{r}\int_\Omega H(x)|u|^r\,dx, \quad \forall u \in X. \tag{1.7}$$