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Interval oscillation criteria for second-order forced impulsive differential equations with mixed nonlinearities ☆

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ABSTRACT

In this paper, by arithmetic–geometric mean inequality and Riccati transformation, interval oscillation criteria are established for second-order forced impulsive differential equation with mixed nonlinearities of the form

$$\begin{cases} \big(r(t)\Phi_\alpha\big(x'(t)\big)\big)' + p_0(t)\Phi_\alpha\big(x(t)\big) + \displaystyle\sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}\big(x(t)\big) = e(t), & t \neq \tau_k, \\ x\big(\tau_k^+\big) = a_k x(\tau_k), \qquad x'\big(\tau_k^+\big) = b_k x'(\tau_k), \end{cases}$$

where $t \geqslant t_0$, $k \in \mathbb{N}$; $\Phi_*(u) = |u|^{*-1}u$; $\{\tau_k\}$ is the impulse moments sequence with $0 \leqslant t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$; $\alpha = p/q$, p, q are odds, and the exponents satisfy

$$\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0.$$

Some known results are generalized and improved. Examples are also given to illustrate the effectiveness and non-emptiness of our results.

1. Introduction

In this paper, we consider the following second-order forced impulsive differential equation with mixed nonlinearities

$$\begin{cases} \big(r(t)\Phi_\alpha\big(x'(t)\big)\big)' + p_0(t)\Phi_\alpha\big(x(t)\big) + \displaystyle\sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}\big(x(t)\big) = e(t), & t \neq \tau_k, \\ x\big(\tau_k^+\big) = a_k x(\tau_k), \qquad x'\big(\tau_k^+\big) = b_k x'(\tau_k), \end{cases} \tag{1.1}$$

where $t \geqslant t_0$, $k \in \mathbb{N}$, $\Phi_*(u) = |u|^{*-1}u$, $\{\tau_k\}$ is the impulse moments sequence with $0 \leqslant t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$, $\lim_{k\to\infty} \tau_k = \infty$, and

$$x(\tau_k) = x\big(\tau_k^-\big) = \lim_{t\to\tau_k - 0} x(t), \qquad x\big(\tau_k^+\big) = \lim_{t\to\tau_k + 0} x(t),$$

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$$x'(\tau_k) = x'(\tau_k^-) = \lim_{h\to 0^-} \frac{x(\tau_k+h)-x(\tau_k)}{h}, \qquad x'(\tau_k^+) = \lim_{h\to 0^+} \frac{x(\tau_k+h)-x(\tau_k^+)}{h}.$$

Throughout this work, we always assume the following conditions hold.

(A_1) $r\in C^1([t_0,\infty),(0,\infty))$, $p_i, e\in C([t_0,\infty),\mathbb{R})$, $i=0,1,\dots,n$;
(A_2) $\beta_1>\cdots>\beta_m>\alpha>\beta_{m+1}>\cdots>\beta_n>0$, $\alpha=p/q$, p, q are odds;
(A_3) $b_k\geqslant a_k>0$, $k\in\mathbb{N}$, are constants.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for oscillation and/or nonoscillation of solutions for different classes of second-order differential equations, see [1]. Great numbers of papers devoted to the particular cases of Eq. (1.1) without impulses such as the linear equation

$$\big(r(t)x'(t)\big)' + q(t)x(t) = 0. \tag{1.2}$$

For instance, the well-known conditions of Fite's type [2], Wintner's type [3], Hartman's type [4], Kamenev's type [5] and Philos' type [6], etc., can guarantee the oscillation of Eq. (1.2). These criteria involve the integral of $q(t)$ hence require information of $q(t)$ on the entire half-line $[t_0,\infty)$. However, from the Sturm Separation Theorem, we see that oscillation is only an interval property, i.e., if there exists a sequence of subintervals $[a_i,b_i]$ of $[t_0,\infty)$, as $a_i\to\infty$, such that for each i there is a nontrivial solution of Eq. (1.2) which has at least two zeros in $[a_i,b_i]$, then every solution of Eq. (1.2) is oscillatory, no matter what the behavior of the coefficients of Eq. (1.2) is on the remaining parts of $[t_0,\infty)$.

Partially applying this idea to oscillation, Kwong and Zettl [7] established a powerful "telescoping principle" that allows us to trim off the troublesome parts of $\int_{t_0}^t q(s)\,ds$ and apply the known criteria to the "good" parts. Unfortunately, this principle requires additional conditions for q on the "bad" parts, i.e., $\int_{b_i}^{a_{i+1}} q(s)\,ds\geqslant 0$, $i\in\mathbb{N}$, which does not reflect the interval oscillation property completely and restricts its applications. For this reason, El-Sayed [8] established an interval criterion for oscillation of a forced second-order equation

$$\big(r(t)x'(t)\big)' + q(t)x(t) = e(t), \tag{1.3}$$

but the result is not very sharp, because a comparison with equations of constant coefficient is used in the proof. In 1999, Wong [9] substantially improved the results of El-Sayed with a more direct and simpler proof. Later, further development of the "interval criteria" for oscillation have been obtained by many authors for both differential equations and delay differential equations in several directions, see [10–16].

Recently, the interval oscillation for a forced mixed type Emden–Fowler equation

$$\big(r(t)x'(t)\big)' + p_0(t)x(t) + \sum_{i=1}^n p_i(t)\Phi_{\beta_i}\big(x(t)\big) = e(t) \tag{1.4}$$

was given much attention because it arise, for instant, in the growth of bacteria population with competitive species. By employing the arithmetic–geometric mean inequality [17] and Riccati technique, Sun and Wong [18], Sun and Meng [19] established interval oscillation theorems for Eq. (1.4) respectively, in which Nasr–Wong oscillation criteria [20,9] were extended.

However, to the best of our knowledge, the study for the oscillation of this mixed type Emden–Fowler equation with impulses is very scare. In 2009, Liu and Xu [21] studied the following forced mixed type impulsive Emden–Fowler equation

$$\begin{cases} \big(r(t)x'(t)\big)' + p_0(t)x(t) + \displaystyle\sum_{i=1}^n p_i(t)\Phi_{\beta_i}\big(x(t)\big) = e(t), \quad t\neq\tau_k, \\ x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \end{cases} \tag{1.5}$$

where $\beta_1>\cdots>\beta_m>1>\beta_{m+1}>\cdots>\beta_n>0$. Employing the arithmetic–geometric mean inequality, they obtained oscillation criteria for Eq. (1.5). In the special case when the impulses are absent, their results extend those of Sun and Wong [18], Sun and Meng [19], Nasr [20], Yang [22], Kong [10], and Wong [9]. Liu and Xu [21] only studied the case of assuming $p_i(t)\geqslant 0$ on considered subinterval $[c_k,d_k]$ for all $i=1,2,\dots,n$. When the restriction on signs of those coefficients corresponding to the sub-linear terms of Eq. (1.5), i.e., $p_i(t)$ for $i=m+1,\dots,n$ was not imposed as in [18,19], no criterion was given by the authors of [21].

Motivated by the ideas in [18,19,21], in this work, we employ the arithmetic–geometric mean inequality, Riccati technique and $\mathcal{H}$ functions (introduced first by Philos [6]) to study the interval oscillation of Eq. (1.1) and its corresponding unforced case

$$\begin{cases} \big(r(t)\Phi_\alpha\big(x'(t)\big)\big)' + p_0(t)\Phi_\alpha\big(x(t)\big) + \displaystyle\sum_{i=1}^n p_i(t)\Phi_{\beta_i}\big(x(t)\big) = 0, \quad t\neq\tau_k, \\ x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k). \end{cases} \tag{1.6}$$