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A system of integral equations on half space

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ABSTRACT

Let R_+^n be the n-dimensional upper half Euclidean space, and let α be any real number satisfying $0 < \alpha < n$, we study positive solutions of the following system of integral equations in R_+^n:

$$\begin{cases} u(x) = \displaystyle\int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) v^q(y)\, dy; \\ v(x) = \displaystyle\int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u^p(y)\, dy \end{cases}$$

where x^* is the reflection of the point x about the plane $x_n = 0$. We assume that $v \in L^{q+1}(R_+^n)$, $u \in L^{p+1}(R_+^n)$ with

$$\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}.$$

In our previous paper, we considered the corresponding single equation

$$u(x) = \int_{R_+^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}} \right) u^{\frac{n+\alpha}{n-\alpha}}(y)\, dy$$

and proved that every positive solution of the above integral equation is rotationally symmetric about some line parallel to x_n-axis. We also established regularity of solutions. Now we go further to study the regularity and rotational symmetry for solutions of the above system of integral equations and generalize the results in our previous paper.

1. Introduction

Let R^n be the n-dimensional Euclidean space, and let α be a real number satisfying $0 < \alpha < n$. Consider the integral equations

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$$\begin{cases} u(x) = \int\limits_{R^n} \frac{1}{|x-y|^{n-\alpha}} v^q(y)\,dy; \\ v(x) = \int\limits_{R^n} \frac{1}{|x-y|^{n-\alpha}} u^p(y)\,dy. \end{cases} \tag{1}$$

Here $\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}$. When $p = q$ and $u(x) = v(x)$, (1) reduces to the following integral equation

$$u(x) = \int\limits_{R^n} \frac{1}{|x-y|^{n-\alpha}} u^\tau(y)\,dy. \tag{2}$$

(2) arises as an Euler–Lagrange equation for a functional under a constraint in the context of the Hardy–Littlewood–Sobolev inequalities (see [13]). In [13], Lieb classified the maximizers of the functional, and thus obtained the best constant in the Hardy–Littlewood–Sobolev inequalities. He then posed the classification of all the critical points of the functional – the solutions of the integral equation (2) as an open problem.

In [9], Chen, Li, and Ou solved Lieb's open problem by using the method of moving planes. They proved that:

Proposition 1. *Every positive regular solution $u(x)$ of* (2) *is radially symmetric and decreasing about some point x_o and therefore assumes the form*

$$c\left(\frac{t}{t^2 + |x-x_0|^2}\right)^{(n-\alpha)/2}, \tag{3}$$

with some positive constants c and t.

They also established the equivalence between the integral equations and the following well-known family of semi-linear partial differential equations

$$(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)}, \quad x \in R^n. \tag{4}$$

In the special case $\alpha = 2$, there have been a series of results concerning the classification of the solutions (cf. [12,1,4,14]). Recently, Wei and Xu [23] generalized these results to the cases that α being any even number between 0 and n. Apparently, for any real value of α between 0 and n, Eq. (4) is also of practical interest and importance.

It can be proved that integral system (1) is equivalent to the PDE system

$$\begin{cases} (-\triangle)^{\frac{\alpha}{2}} u(x) = v^q(x), & \forall x \in R^n; \\ (-\triangle)^{\frac{\alpha}{2}} v(x) = u^p(x), & \forall x \in R^n. \end{cases}$$

In the special case when $\alpha = 2$, it reduces to the well-known Lane–Emden system.

On the upper half Euclidean space

$$R^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in R^n_+ \mid x_n > 0\}.$$

The same equation naturally arises with Navier boundary conditions. In particular, when α is an even number, we have

$$\begin{cases} (-\triangle)^{\frac{\alpha}{2}} u(x) = v^q(x), & \forall x \in R^n_+; \\ (-\triangle)^{\frac{\alpha}{2}} v(x) = u^p(x), & \forall x \in R^n_+; \\ (-\triangle)^k u(x) = 0, & \forall x \in \partial R^n_+; \\ (-\triangle)^k v(x) = 0, & \forall x \in \partial R^n_+, \end{cases} \tag{5}$$

$k = 0, 1, \ldots, \frac{\alpha}{2} - 1$.

In this paper, we will study the corresponding system of integral equations in the upper half space R^n_+,

$$\begin{cases} u(x) = \int\limits_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right) v^q(y)\,dy; \\ v(x) = \int\limits_{R^n_+} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^*-y|^{n-\alpha}}\right) u^p(y)\,dy. \end{cases} \tag{6}$$

One of our motivation is

Theorem 1. *Let $(u(x), v(x))$ be a pair of positive smooth solutions of* (6)*, then $(u(x), v(x))$ satisfies* (5)*.*