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Periodic solutions for Hamiltonian systems without Ambrosetti–Rabinowitz condition and spectrum 0 ☆

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ABSTRACT

In this paper, we consider the superquadratic second order Hamiltonian system

$$u''(t)+A(t)u(t)+\nabla H\big(t,u(t)\big)=0,\quad t\in\mathbb{R}.$$

Our main results here allow the classical Ambrosetti–Rabinowitz superlinear condition to be replaced by a general superquadratic condition, and 0 lies in a gap of $\sigma(B)$, where $B:=-\frac{d^2}{dt^2}-A(t)$. We will study the ground state periodic solutions for this problem. The main idea here lies in an application of a variant generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou.

1. Introduction and main results

Let us consider the following second order Hamiltonian system

$$u''(t)+A(t)u(t)+\nabla H\big(t,u(t)\big)=0,\quad t\in\mathbb{R},\tag{1.1}$$

where $A(\cdot)$ is a continuous T-periodic symmetric matrix, $H:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is T-periodic ($T>0$) in its first variable. Moreover, we always assume that $H(t,x)$ is continuous in t for each $x\in\mathbb{R}^N$, continuously differentiable in x for each $t\in[0,T]$ and $\nabla H(t,x)$ denotes its gradient with respect to the x variable.

Hamiltonian systems are physical systems in which forces are momentum invariant in classical mechanics, and they are studied in Hamiltonian mechanics. Hamiltonian systems are systems of differential equations which can be written in the form of Hamilton's equations in mathematics, and they are usually formulated in terms of Hamiltonian vector fields on a symplectic manifold or Poisson manifold. Hamiltonian systems are a special case of dynamical systems. The study of gas dynamics, fluid mechanics, relativistic mechanics and nuclear physics is very important.

Rabinowitz [20] established the existence of periodic solutions of (1.1) with $A(t)=0$ under the following superquadratic condition: there exist a constant $\mu>2$ and $L>0$ such that

$$0<\mu H(t,x)\leqslant\big(\nabla H(t,x),x\big),\quad \forall|x|\geqslant L,\ t\in[0,T],\tag{1.2}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$. It is known as Ambrosetti–Rabinowitz superquadratic condition (AR-condition for short). As we known that the AR-condition is very convenient in checking the mountain pass geometry and verifying the Palais–Smale condition (PS-condition), for the associated Euler functional. Since then, this condition has been used extensively in many literatures, see [1–3,5,7–9,14,15,21] and references therein.

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There have been some authors tried to remove the AR-condition. By using the linking theorem, Fei [6] proved an existence theorem of (1.1) with $A(t) = 0$ for all $t \in [0, T]$ and a superquadratic potential function $H \in C^1([0, T], \mathbb{R}^N)$, where H does not satisfy the AR-condition. He and Wu [11], Luan and Mao [16] and Tao, Yan and Wu [26] studied (1.1) with $A(t) \not\equiv 0$ by the local linking theorem (see [15]), and employed additional restrictive conditions, such as the positivity of the potential function H, superquadratic behavior near the origin or additional growth restrictions near infinity (see, for example, hypothesis H_3' in [16]). We should also mention that some authors studied systems with nonsmooth, locally Lipschitz potentials (hemivariational inequalities), see [4,18,19]. Barletta and Papageorgiou [4] assumed that the linearization of (1.1) with a trivial negative part and used a nonsmooth version of the local linking theorem (see [13]). By using nonsmooth critical point theory (see [10]), Motreanu and his partners [18] considered superquadratic systems and used a nonsmooth version of the AR-condition, and the authors [19] considered systems with an indefinite linear part and assumed that the potential function was subquadratic near infinity.

In this paper, our approach is based on an application of a variant generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou [24], see also [23,28], where the authors developed the idea of Monotonicity Trick for strongly indefinite problems, the original idea is due to [12,22]. To the best of our knowledge, this is the first work on superlinear second order periodic Hamiltonian systems by this method. Here, the classical AR-condition on ∇H is replaced by a general superquadratic condition, and we are interested in the existence of ground state period solutions of (1.1), that is, solutions corresponding to the least energy of the action functional of (1.1). There are some results concerning the existence of ground state solutions for Schrödinger equations, see [25,31]. We should mention that some authors have studied several different problems by the same method as our paper, see [29–31] and their references therein. Among these problems are discrete Schrödinger equation with spectrum zero [29], Schrödinger equation with spectrum zero [30] and Schrödinger equation without spectrum zero [31].

In this paper, we assume that 0 lies in a gap of $\sigma(B)$, where $B := -\frac{d^2}{dt^2} - A(t)$, that is,

(L_0) $\underline{\Lambda} := \sup(\sigma(B) \cap (-\infty, 0)) < 0 < \overline{\Lambda} := \inf(\sigma(B) \cap (0, \infty))$.

To state our main result, we still need the following assumptions:

(H_0) $(\nabla H(t, u), v)(u, v) \geqslant 0$ uniformly in t.
(H_1) $|\nabla H(t, u)| \leqslant a(1 + |u|^{p-1})$ for some $a > 0$ and $p > 2$.
(H_2) $|\nabla H(t, u)| = o(|u|)$ as $|u| \to 0$ uniformly in t.
(H_3) $\frac{H(t,u)}{|u|^2} \to \infty$ as $|u| \to \infty$ uniformly in t.
(H_4) $H(t, u) \geqslant 0$ for all $u \in \mathbb{R}^N$, $\frac{1}{2}(\nabla H(t, u), u) > H(t, u)$ for all $u \in \mathbb{R}^N \backslash \{0\}$.
(H_5) $H(t, u) = H(t, v)$ and $(\nabla H(t, u), v) \leqslant (\nabla H(t, u), u)$ uniformly in t, if $|u| = |v|$.
(H_6) $(\nabla H(x, u), v) \neq (\nabla H(t, v), u)$ for any $t \in \mathbb{R}$, if $|u| \neq |v|$ and $(u, v) \neq 0$.

As is shown in next examples, our assumptions are reasonable and there are cases in which the well-known Ambrosetti–Rabinowitz superquadratic condition is not satisfied.

Example 1.1. Let $H(t, u) = |u|^p$, where $p > 2$. Clearly, $H(t, u)$ satisfies (H_0)–(H_6) and the well-known Ambrosetti–Rabinowitz superquadratic condition.

Example 1.2. Let $H(t, u) = g(t)(|u|^p + (p - 2)|u|^{p-\varepsilon} \sin^2(|u|^\varepsilon/\varepsilon))$, where $g(t) > 0$ is T-periodic in t, $0 < \varepsilon < p - 2$ and p is the parameter in (H_1). Note that

$$\nabla H(t, u) = g(t)u\left[(p-2)(p-\varepsilon)|u|^{p-\varepsilon-2} \sin^2\left(\frac{|u|^\varepsilon}{\varepsilon}\right) + \left(p + (p-2)\sin\left(\frac{2|u|^\varepsilon}{\varepsilon}\right)\right)|u|^{p-2}\right].$$

It is not hard to check that $H(t, u)$ satisfies (H_0)–(H_6) but does not satisfy the Ambrosetti–Rabinowitz superquadratic condition.

In the present paper, we study the existence of ground state period solutions for the second order Hamiltonian system (1.1). Now, our main result reads as follows:

Theorem 1.1. *If assumptions (L_0) and (H_0)–(H_6) are satisfied, then* (1.1) *has at least one ground state T-periodic solution.*

The rest of the present paper is organized as follows. In Section 2, we establish the variational framework associated with (1.1), and we also give some preliminary lemmas, which are useful in the proof of our main result. In Section 3, we give the detailed proof of our main result.