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Global existence and blow-up solutions for doubly degenerate parabolic system with nonlocal source

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ABSTRACT

This paper deals with the following nonlocal doubly degenerate parabolic system

$$u_t - \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) = a\int_\Omega u^{\alpha_1}(x,t)v^{\beta_1}(x,t)\,dx,$$

$$v_t - \operatorname{div}\left(\left|\nabla v^n\right|^{q-2}\nabla v^n\right) = b\int_\Omega u^{\alpha_2}(x,t)v^{\beta_2}(x,t)\,dx$$

with null Dirichlet boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, where $m,n \geqslant 1$, $p,q > 2$, $\alpha_i, \beta_i \geqslant 0$, $i = 1,2$ and $a,b > 0$ are positive constants. We first get the non-existence result for a related elliptic systems of non-increasing positive solutions. Secondly by using this non-existence result, blow-up estimates for above non-Newton polytropic filtration systems with the homogeneous Dirichlet boundary value conditions are obtained. Then under appropriate hypotheses, we establish local theory of the solutions and prove that the solution either exists globally or blows up in finite time. In the special case, $\beta_1 = n(q-1) - \beta_2$, $\alpha_2 = m(p-1) - \alpha_1$, we also give a criterion for the solution to exist globally or blow up in finite time, which depends on a, b and $\zeta(x)$, $\vartheta(x)$ as defined in the main results.

1. Introduction

In this paper, we consider the following nonlocal doubly degenerate parabolic system,

$$\begin{aligned}
&u_t - \Delta_{m,p} u = a\int_\Omega u^{\alpha_1}(x,t)v^{\beta_1}(x,t)\,dx, \quad (x,t) \in \Omega_T,\\
&v_t - \Delta_{n,q} v = b\int_\Omega u^{\alpha_2}(x,t)v^{\beta_2}(x,t)\,dx, \quad (x,t) \in \Omega_T,\\
&u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T],\\
&u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \Omega,
\end{aligned} \tag{1.1}$$

where for $k > 0$, $\gamma > 2$ and $N \geqslant 1$,

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$$\Delta_{k,\gamma}\Theta = \nabla\cdot\left(\left|\nabla\Theta^k\right|^{\gamma-2}\cdot\nabla\Theta^k\right), \qquad \nabla\Theta^k = k\Theta^{k-1}(\Theta_{x_1},\ldots,\Theta_{x_N}),$$

$\Omega\subset\mathbb{R}^N$ ($N\geqslant 1$) is a bounded domain with appropriately smooth boundary $\partial\Omega$; $m,n\geqslant 1$, $p,q>2$, $\alpha_i,\beta_i\geqslant 0$, $i=1,2$, $\Omega_T=\Omega\times(0,T]$ and a, b are positive constants and u_0, v_0 satisfy compatibility and the following conditions:

$$(H)\quad u_0^m\in C(\overline{\Omega})\cap W_0^{1,p}(\Omega),\qquad v_0^n\in C(\overline{\Omega})\cap W_0^{1,q}(\Omega)\quad\text{and}\quad \nabla u_0^m\cdot\nu<0,$$
$$\nabla v_0^n\cdot\nu<0\quad\text{on }\partial\Omega,\ \text{where } \nu \text{ is unit outer normal vector on } \partial\Omega.$$

Parabolic systems like (1.1) arise in many applications in the fields of mechanics, physics and biology like, for instance, the description of turbulent filtration in porous media, the theory of non-Newtonian fluids perturbed by nonlinear terms and forced by rather irregular period in time excitations, the flow of a gas through a porous medium in a turbulent regime or the spread of biological (see [1–4] and the references given therein); in general, doubly nonlinear parabolic equations are used to model processes obeying a nonlinear Darcy law (see [5,6] and the references given therein). In the non-Newtonian fluids theory, the pair (p,q) is a characteristic quantity of medium. Media with $(p,q)>(2,2)$ are called dilatant fluids and those with $(p,q)<(2,2)$ are called pseudo-plastics. If $(p,q)=(2,2)$, they are called Newtonian fluids. When $(p,q)=(2,2)$ and $(m,n)>(1,1)$ the connection with the flow in porous media is by now classical. When $(m,n)\geqslant(1,1)$ and $(p,q)>(2,2)$, the system models the non-stationary, polytropic flow of a fluid in a porous medium whose tangential stress has a power dependence on the velocity of the displacement under polytropic conditions (non-Newtonian elastic filtration); it has been intensively studied (see [7–9] and references therein). We refer to [10] for further information on these phenomena. Recently a connection has been revealed with soil science; specifically with flows in reservoirs exhibiting fractured media (see [11]).

The problems with nonlinear reaction term and nonlinear diffusion include blow-up and global existence conditions of solutions, blow-up rates and blow-up sets, etc. (see the surveys [12,13]). Here, we say solution blows up in finite time if the solution becomes unbounded (in the sense of maximum norm) at that time. System (1.1) has been studied by many authors. For $p=q=2$, $m=n=1$, it is a classical reaction–diffusion system of Fujita type (see [14,15] for nonlinear boundary conditions, see [16] for local nonlinear reaction terms, see [9,17] for nonlocal nonlinear reaction terms).

In the last three decades, many authors have studied the following degenerate parabolic problem:

$$\begin{aligned}
&u_t-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=f(u),\quad (x,t)\in\Omega\times(0,T],\\
&u(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T],\\
&u(x,0)=u_0(x),\quad x\in\Omega
\end{aligned}\tag{1.2}$$

under different conditions. In [1,17–23], the existence, uniqueness, extinction phenomenon and regularity of solutions were obtained. If $f(u)=u^q$, $q>1$, the results in [14,24–27] read: (1) the solution u exists globally if $q<p-1$; (2) u blows up in finite time if $q>p-1$ and $u_0(x)$ is sufficiently large. Li and Xie [17] studied the following Eq. (1.2) with $f(u)=\int_\Omega u^q(x,t)\,dx$ under null Dirichlet conditions and obtained that the solution either exists globally or blows up in finite time. Under appropriate hypotheses, they had local theory of the solution and obtained that the solution either exists globally or blows up in finite time.

Li et al. in [28] deal with the following reaction–diffusion system:

$$\begin{aligned}
&u_t-\Delta u=\int_\Omega f\left(v(y,t)\right)dy,\quad x\in\Omega,\ t>0,\\
&v_t-\Delta v=\int_\Omega g\left(u(y,t)\right)dy,\quad x\in\Omega,\ t>0
\end{aligned}$$

with initial and boundary conditions. They proved that there exists a unique classical solution and the solution either exists globally or blows up in finite time. Furthermore, they obtained the blow-up set and asymptotic behavior provided that the solution blows up in finite time.

For p-Laplacian systems, Cui and Yang [29] and Li [30] studied the following equations:

$$\begin{aligned}
&u_t-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=\int_\Omega v^\alpha\,dx,\quad (x,t)\in\Omega\times(0,T],\\
&v_t-\operatorname{div}\left(|\nabla v|^{q-2}\nabla v\right)=\int_\Omega u^\beta\,dx,\quad (x,t)\in\Omega\times(0,T],\\
&u(x,t)=v(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T],\\
&u(x,0)=u_0(x),\qquad v(x,0)=v_0(x),\quad x\in\Omega,
\end{aligned}\tag{1.3}$$

which derive some estimates near the blow-up point for positive solutions and non-existence of positive solutions of the relate elliptic systems, with global existence and blow-up of solutions for (1.3).