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A remark on Schatten–von Neumann properties of resolvent differences of generalized Robin Laplacians on bounded domains

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ABSTRACT

In this note we investigate the asymptotic behavior of the s-numbers of the resolvent difference of two generalized self-adjoint, maximal dissipative or maximal accumulative Robin Laplacians on a bounded domain Ω with smooth boundary $\partial\Omega$. For this we apply the recently introduced abstract notion of quasi boundary triples and Weyl functions from extension theory of symmetric operators together with Krein type resolvent formulae and well-known eigenvalue asymptotics of the Laplace–Beltrami operator on $\partial\Omega$. It is shown that the resolvent difference of two generalized Robin Laplacians belongs to the Schatten–von Neumann class of any order p for which

$$p > \frac{\dim\Omega - 1}{3}.$$

Moreover, we also give a simple sufficient condition for the resolvent difference of two generalized Robin Laplacians to belong to a Schatten–von Neumann class of arbitrary small order. Our results extend and complement classical theorems due to M.Š. Birman on Schatten–von Neumann properties of the resolvent differences of Dirichlet, Neumann and self-adjoint Robin Laplacians.

1. Introduction

It is well known that the difference of the resolvents of two self-adjoint extensions of a symmetric operator (with equal infinite deficiency numbers) in many cases behaves 'better' than the resolvents themselves, e.g. even if the resolvents are non-compact operators, the difference may belong to a Schatten–von Neumann class, or if the resolvents are from a Schatten–von Neumann class, the difference may lie in one of smaller order. In particular, according to classical results due to M.Š. Birman [6] the resolvent difference of the Dirichlet and Neumann Laplacian in a bounded or unbounded domain Ω with compact C^∞ boundary $\partial\Omega$ satisfies

$$\left(-\Delta_D^\Omega - \lambda\right)^{-1} - \left(-\Delta_N^\Omega - \lambda\right)^{-1} \in \mathcal{S}_p\left(L^2(\Omega)\right) \quad \text{for all } p > \frac{\dim\Omega - 1}{2},$$

where $\mathcal{S}_p(L^2(\Omega))$ is the Schatten–von Neumann class of order p and Δ_D^Ω, Δ_N^Ω are the Dirichlet and Neumann Laplacians on Ω, respectively. Analogous estimates were also obtained for the difference of the resolvents of self-adjoint Laplacians

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with (ordinary) Robin boundary conditions $\beta f|_{\partial\Omega} = \frac{\partial f}{\partial\nu}$, where β is a real-valued function on $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the outer normal derivative. Later such results on spectral asymptotics were refined and generalized by, e.g. M.Š. Birman and M.Z. Solomjak in [7] and G. Grubb in [19]. Recently some new Schatten–von Neumann properties of resolvent differences of differential operators were announced by F. Gesztesy and M.M. Malamud in [13] and [25], and in the paper by G. Grubb [23] the influence of generalized Robin boundary conditions on the essential spectrum in exterior domains was studied.

The main objective of the present paper is to extend and complement some results on Schatten–von Neumann properties for the resolvent difference of self-adjoint Laplacians from [6]. Instead of Dirichlet, Neumann and self-adjoint Robin Laplacians we study so-called *generalized* Robin Laplacians which are self-adjoint, maximal dissipative or maximal accumulative. More precisely, we study self-adjoint, maximal dissipative and maximal accumulative realizations $-\Delta^{\Omega}_{\Theta_1}$ and $-\Delta^{\Omega}_{\Theta_2}$ of the Laplacian corresponding to the generalized (or non-local) Robin boundary conditions

$$\Theta_1 \frac{\partial f}{\partial\nu}\bigg|_{\partial\Omega} = f|_{\partial\Omega} \quad \text{and} \quad \Theta_2 \frac{\partial f}{\partial\nu}\bigg|_{\partial\Omega} = f|_{\partial\Omega},$$

respectively, where Θ_1 and Θ_2 are self-adjoint, maximal dissipative or maximal accumulative operators in $L^2(\partial\Omega)$ such that $0 \notin \sigma_{\text{ess}}(\Theta_i)$, $i = 1, 2$. We note that generalized self-adjoint Robin Laplacians were recently also considered by F. Gesztesy and M. Mitrea in [14–17]. It is shown in Theorem 3.5 and Corollary 3.6 that

$$\left(-\Delta^{\Omega}_{\Theta_1} - \lambda\right)^{-1} - \left(-\Delta^{\Omega}_{\Theta_2} - \lambda\right)^{-1} \in \mathcal{S}_p\left(L^2(\Omega)\right) \quad \text{for all } p > \frac{\dim\Omega - 1}{3} \tag{1.1}$$

holds for all $\lambda \in \rho(-\Delta^{\Omega}_{\Theta_1}) \cap \rho(-\Delta^{\Omega}_{\Theta_2})$. Moreover, if $\Theta_1 - \Theta_2 \in \mathcal{S}_{p_0}(L^2(\Omega))$ for some $p_0 \in (0, \infty)$, then

$$\left(-\Delta^{\Omega}_{\Theta_1} - \lambda\right)^{-1} - \left(-\Delta^{\Omega}_{\Theta_2} - \lambda\right)^{-1} \in \mathcal{S}_p\left(L^2(\Omega)\right) \quad \text{for all } p > \frac{(\dim\Omega - 1)p_0}{(\dim\Omega - 1) + 3p_0}; \tag{1.2}$$

see Theorem 3.11. The proofs of these estimates are quite elementary and short when applying the abstract concept of quasi boundary triples and Weyl functions from extension theory of symmetric operators together with Krein type resolvent formulae from [5] and well-known eigenvalue asymptotics of the Laplace–Beltrami operator on $\partial\Omega$; see, e.g. [2]. We note that our main results (1.1) and (1.2) can be proved in the same way for generalized Robin Schrödinger operators $-\Delta^{\Omega}_{\Theta_i} + V$ with a real valued L^∞ potential V or for more general uniformly elliptic differential operators with coefficients satisfying appropriate conditions.

2. Quasi boundary triples

In this section we briefly recall the abstract notion of quasi boundary triples and Weyl functions in extension theory of symmetric operators, some of their properties and how they can be applied to the Laplacian on bounded domains. This concept was introduced in connection with elliptic boundary value problems by the first two authors in [5] as a generalization of the notion of ordinary and generalized boundary triples from [9–12,24]. The following definition is a variant of [5, Definition 2.1] for densely defined, closed, symmetric operators.

Definition 2.1. Let A be a densely defined, closed, symmetric operator in a Hilbert space $\mathcal{H}$. We say that $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is a *quasi boundary triple* for A^* if $\mathcal{G}$ is a Hilbert space, Γ_0 and Γ_1 are linear mappings defined on the same subset $\operatorname{dom}\Gamma_0 = \operatorname{dom}\Gamma_1$ of $\operatorname{dom} A^*$ with values in $\mathcal{G}$ such that $T := A^*|_{\operatorname{dom}\Gamma_0}$ satisfies $\overline{T} = A^*$, that $\binom{\Gamma_0}{\Gamma_1}: \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ has dense range, that $A_0 := T|_{\ker\Gamma_0}$ is self-adjoint and that the identity

$$(Tf, g)_{\mathcal{H}} - (f, Tg)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$$

holds for all $f, g \in \operatorname{dom} T$.

From the definition it follows that both $\operatorname{ran}\Gamma_0$ and $\operatorname{ran}\Gamma_1$ are dense in $\mathcal{G}$. Moreover, one can easily show that $\Gamma_0|_{\ker(T-\lambda)}$ is bijective from $\ker(T - \lambda)$ onto $\operatorname{ran}\Gamma_0$ for $\lambda \in \rho(A_0)$. Next we recall the definition of the γ-field, the Weyl function and the parameterization of certain extensions of the symmetric operator A.

Definition 2.2. Let A be a densely defined, closed, symmetric operator in a Hilbert space, $(\mathcal{G}, \Gamma_0, \Gamma_1)$ a quasi boundary triple for A^* and T as above.

(i) The bijective mapping

$$\gamma(\lambda) := \left(\Gamma_0|_{\ker(T-\lambda)}\right)^{-1} : \operatorname{ran}\Gamma_0 \to \ker(T - \lambda), \quad \lambda \in \rho(A_0),$$

is called *γ-field*.