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Periodic solutions of a logistic type population model with harvesting ☆

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ABSTRACT

We consider a bifurcation problem arising from population biology

$$\frac{du(t)}{dt} = f\big(u(t)\big) - \varepsilon h(t),$$

where $f(u)$ is a logistic type growth rate function, $\varepsilon \geqslant 0$, $h(t)$ is a continuous function of period T such that $\int_0^T h(t)\,dt > 0$. We prove that there exists an $\varepsilon_0 > 0$ such that the equation has exactly two T-periodic solutions when $0 < \varepsilon < \varepsilon_0$, exactly one T-periodic solution when $\varepsilon = \varepsilon_0$, and no T-periodic solution when $\varepsilon > \varepsilon_0$.

1. Introduction

When a population grows at a density-dependent growth rate $f(u)$ and it is harvested with a seasonal harvesting rate $h(t)$ with period T, the population can be described by a differential equation (see for example, [3,4,8])

$$\frac{du(t)}{dt} = f\big(u(t)\big) - h(t). \tag{1.1}$$

Here we assume that the non-linear function f is a logistic type function which satisfies

(f1) $f \in C^2(\mathbf{R})$, $f(0) = 0$, $f'(0) > 0$, $f(u) > 0$ for $u \in (0, M)$, $f(M) = 0$ and $f'(M) < 0$;
(f2) $f''(u) < 0$ for $u \in \mathbf{R}$.

Some typical examples of $f(u)$ are $f(u) = au - bu^p$, where $a, b > 0$, $p \geqslant 2$, see [10,12,17,20]. When $h(t)$ is a constant h, then it is easy to know that there is a threshold (maximum sustainable yield) $h_* > 0$ such that when $h > h_*$, (1.1) has no equilibrium and the population is destined to extinction, and when $h < h_*$, there are exactly two positive equilibria. When the seasonal effect on the harvesting is considered ($h(t)$ periodic), then one expects that periodic solutions play similar role as equilibria in the constant case, and the question is: how many periodic solutions does (1.1) have?

Here we assume that the total yield over one season (period) is positive, that is $\int_0^T h(t)\,dt > 0$. Thus we allow $h(t)$ to be negative, that is stocking instead of harvesting, but the total effort is still harmful to the population. Without loss of

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generality, we can normalize $h(t)$ so that $\int_0^T h(t)\,dt = T$, and we rewrite (1.1) to be

$$\frac{du(t)}{dt} = f\big(u(t)\big) - \varepsilon h(t), \tag{1.2}$$

where $\varepsilon \in \mathbf{R}$ measures the harvesting strength. Our result is

Theorem 1.1. *Suppose that f satisfies* (f1) *and* (f2)*. Let $h(t)$ be a continuous function of period T such that $\int_0^T h(t)\,dt = T$. Then there exists an $\varepsilon_0 > 0$ such that* (1.2) *has exactly two T-periodic solutions when $\varepsilon < \varepsilon_0$, exactly one T-periodic solution when $\varepsilon = \varepsilon_0$, and no T-periodic solution when $\varepsilon > \varepsilon_0$.*

Thus the dynamics of (1.2) is qualitatively similar to the autonomous equation with constant $h(t)$, with two equilibria replaced by two periodic solutions. One can also define ε_0 as the maximum sustainable yield in this case.

It is known that (1.2) has at most two periodic solutions due to the concavity of f, see Pliss [18], Lazer and Sànchez [13], Mawhin [15], and Korman and Ouyang [11]. The turning point (fold) structure is also studied in [11,15], as well as McKean and Scovel [16]. But the main result in [11,16] is for a more general problem, and the result is abstract in describing the singular points. The result in [15] assumes that f depends on t, but $h(t)$ is assumed to be a constant or strictly positive (see [15, Remark 2]). Our result here is more specific in term of harvesting model, and it is more general than the one in [15] since we only assume that $\int_0^T h(t)\,dt > 0$. Our proof uses some ingredients in previous approach, but also some more recent bifurcation theory. A different approach was given in Benardete, Noonburg and Pollina [3], based on Poincaré map and dynamical systems arguments, and they proved a special case of Theorem 1.1 when $f(x) = Rx(1-x)$ and $h(t) = 1 + \alpha\sin(2\pi t)$. Other recent discussions can be found in [2,5–7], for example.

We give preliminaries in Section 2, and we prove the main result in Section 3. Some discussions, numerical examples and conjectures are given in Section 4. An earlier version of Theorem 1.1 appeared in Problem Section of Electronic Journal of Differential Equations in 2006 [19].

2. Preliminaries

To prove the theorem we recall the following result based on the implicit function theorem:

Lemma 2.1. *Consider*

$$x' = f(\varepsilon, t, x), \tag{2.1}$$

where $f \in C^1(\mathbf{R} \times \mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$, and $x \in \mathbf{R}^n$. We suppose that $f(\varepsilon, t+T, x) = f(\varepsilon, t, x)$ for all $(\varepsilon, t, x) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n$, and for $\varepsilon = 0$, (2.1) *has a T-periodic solution $y = y(t)$. Let $z(\varepsilon, t, \xi)$ be the solution of the initial value problem*:

$$z' = f(\varepsilon, t, z), \quad t > 0, \qquad z(0) = \xi, \tag{2.2}$$

and let $A(\varepsilon, t, \xi) = \partial z(\varepsilon, t, \xi)/\partial \xi$. Suppose that $\lambda = 1$ is not an eigenvalue of $A(0, T, y(0))$. Then there exists a $\delta > 0$ such that for $|\varepsilon| < \delta$, there exists a C^1 function $\xi(\varepsilon)$ such that $\xi(0) = y(0)$, and (2.1) *has a unique T-periodic solution $y_\varepsilon(t)$ with $y_\varepsilon(0) = \xi(\varepsilon)$.*

Proof. This lemma is well known, see for example, [1]. For the sake of completeness, we include the proof here. Notice that a T-periodic solution satisfies $z(\varepsilon, T, \xi) = \xi$. Define $F : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ by $F(\varepsilon, \xi) = z(\varepsilon, T, \xi) - \xi$. Then F is continuously differentiable, $F(0, y(0)) = 0$, $F_\xi(0, y(0)) = A(0, T, y(0)) - I$. Since $\lambda = 1$ is not an eigenvalue of $A(0, T, y(0))$, then $F_\xi(0, y(0))$ is invertible, and the claimed result follows from the implicit function theorem. □

We also recall a well-known result for concave non-linearity. A particular case of Lemma 2.2 was known in Pliss [18], and the current version is due to Mawhin [15] (see also Korman and Ouyang [11]).

Lemma 2.2.

$$x' = f(t, x), \tag{2.3}$$

where $f(t+T, x) = f(t, x)$ and $f_{xx}(t, x) < 0$ for all $(t, x) \in \mathbf{R} \times \mathbf{R}^n$. Then (2.3) *has at most two T-periodic solutions.*

We also recall the following well-known bifurcation theorem in [9] and a new bifurcation theorem of the authors [14].

Theorem 2.3 *(Saddle-node bifurcation theorem of Crandall and Rabinowitz [9]). Suppose that X and Y are Banach spaces. Let $(\lambda_0, u_0) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y. Suppose that*