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Solutions of a class of Hamiltonian elliptic systems in R^N

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ABSTRACT

We study the existence of ground state solutions for the following elliptic systems in R^N

$$\begin{cases} -\Delta u + b\cdot\nabla_x u + u = H_v(x,u,v), \\ -\Delta v - b\cdot\nabla_x v + v = H_u(x,u,v), \end{cases}$$

where $b=(b_1,\dots,b_N)$ is a constant vector and $H\in C^1(R^N\times R^2,R)$ is nonperiodic in variables x and super-quadratic as $|z|\to\infty$. By a recent critical point theorem for strongly indefinite problem, we obtain the existence of at least one ground state solution.

1. Introduction and main results

In this paper we are going to study the following Hamiltonian elliptic systems

$$\begin{cases} -\Delta u + b\cdot\nabla_x u + u = H_v(x,u,v), \\ -\Delta v - b\cdot\nabla_x v + v = H_u(x,u,v), \end{cases} \tag{H.S}$$

where $b=(b_1,\dots,b_N)$ is a constant vector and $H\in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$. We are interested in the existence of solutions with least energy.

As we know, existence or non-existence of solutions of Hamiltonian elliptic systems has been a subject of active research in recent years. When $b=0$, $H(x,u,v)=F(x,u)+G(x,v)$, the system $(H.S)$ becomes the following problem

$$\begin{cases} -\Delta u + u = g(x,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + v = f(x,u) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.1}$$

de Figueiredo and Yang [13] showed the existence of a strong radial solution pair for (1.1) under the assumptions that $f(x,t)$ and $g(x,t)$ (satisfying Ambrosetti–Rabinowitz condition) are superlinear and radially symmetric with respect to x, $|f(x,t)|\leqslant c(1+|t|^p)$, $|g(x,t)|\leqslant c(1+|t|^q)$ with $2\leqslant p,q<2N/(N-2)$, $N>2$. This result was later generalized by Sirakov in [22], there the author considered

$$\begin{cases} -\Delta u + b(x)u = g(x,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + b(x)v = f(x,u) & \text{in } \mathbb{R}^N \end{cases} \tag{1.2}$$

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with f, g satisfying the above growth condition and p, q satisfying

$$\frac{1}{p}+\frac{1}{q}>\frac{N-2}{N}. \tag{1.3}$$

Recently, Li and Yang [18] considered (1.1) with asymptotically linear nonlinearities, there the authors obtained the existence of positive solution by using a generalized linking theorem for strongly indefinite problem. In [2], by using the dual variational method, Alves et al. investigated the existence of positive solutions for

$$\begin{cases} -\Delta u+u=W_1(x)|v|^{p-1}v & \text{in } \mathbb{R}^N, \\ -\Delta v+v=W_2(x)|u|^{q-1}u & \text{in } \mathbb{R}^N \end{cases} \tag{1.4}$$

with asymptotically periodic nonlinearities and p, q satisfying (1.3).

There are also several papers considering elliptic systems of Hamiltonian type on bounded domain Ω,

$$\begin{cases} -\Delta u=g(x,v) & \text{in } \Omega, \\ -\Delta v=f(x,u) & \text{in } \Omega, \\ u=v=0 & \text{in } \partial\Omega. \end{cases}$$

Hulshof and Van der Vorst [17] and de Figueiredo and Felmer [15] considered this elliptic systems under condition (1.6) by using the Sobolev spaces of fractional order.

When $b \neq 0$, there are not much work on elliptic systems with gradient terms. de Figueiredo and Ding [12] considered the following Hamiltonian type systems

$$\begin{cases} -\Delta u+b(x)\cdot\nabla_x u+V(x)u=H_v(x,u,v), \\ -\Delta v-b(x)\cdot\nabla_x v+V(x)v=H_u(x,u,v), \end{cases} \tag{1.5}$$

where $b(x)$ is a vector with gauge condition ($div\, b(x)=0$), $V(x)$ and even nonlinearity $H(x,u,v)$ are assumed to be periodic, the authors obtained the existence of infinitely many nontrivial solutions.

In the present paper we are interested in the existence of ground state solutions for Hamiltonian elliptic systems with nonperiodic superlinear nonlinearities. Since the system $(H.S)$ is of Hamiltonian type in $\mathbb{R}^N$, we need to overcome some difficulties. First, the problem is set in $\mathbb{R}^N$, there is no compactness for the Sobolev imbedding. Second, the energy functional is strongly indefinite, the classical critical point theory cannot be applied directly. Third, the nonlinearities $H(x,u,v)$ are nonperiodic in variable x and superlinear at infinity, the method in [12] cannot be applied to obtain the existence of solutions. Moreover the appearance of the gradient terms in the systems also brings us some difficulties, in this case, the variational framework in [2,18] cannot work any longer. Inspired by recent works of Ding and Wei [10], Ding and Lee [25] and Li and Yang [18], We are going to investigate the existence of ground state solutions for the Hamiltonian elliptic systems $(H.S)$. In [10] the authors considered a class of nonlinear Dirac equations with nonperiodic potential and superlinear nonlinearities. They obtained the existence of ground state solutions by reduction methods. There they also give the exponential decaying proposition for those solutions. For other results about Hamiltonian system and strongly indefinite problem we refer readers to [5,8,9,11,16].

To continue the discussion, we need the following notations

$$\mathcal{J}=\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad \mathcal{J}_0=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \mathcal{S}=-\Delta+1, \quad z=(u,v)$$

and

$$A=\mathcal{J}_0\mathcal{S}+\mathcal{J}b\cdot\nabla.$$

Then $(H.S)$ can be rewritten as

$$Az=H_z(x,z).$$

It was called an unbounded Hamiltonian system [23], or an infinite-dimensional Hamiltonian system [3], which can also be obtained from the diffusion system, see [6,20]. We make the following assumptions:

(H_1) $H_z(x,z)=h(x,|z|)z$, $h(x,s)\geqslant 0$, $h(x,s)=o(s)$ as $s\to 0$.

(H_2) There is $r>0$ and $1<q<\frac{2}{\sigma-1}$ with $\sigma>\max\{1,\frac{N}{2}\}$, such that

$$H(x,z)\geqslant C_0|z|^{q+1},$$
$$\left|h\big(x,|z|\big)\right|^{\sigma}\leqslant C_1\tilde{H}(x,z), \quad \text{if } |z|\geqslant r,$$

where $\tilde{H}(x,z)=\frac{1}{2}h(x,|z|)|z|^2-H(x,z)>0$, if $z\neq 0$.

(H_3) There is $h_\infty\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ with $h'_\infty(s)>0$ for $s>0$ such that $h(x,s)\to h_\infty(s)$ as $|x|\to\infty$ uniformly on bounded sets of s, and $h_\infty(s)<h(x,s)$ for all (x,s). Moreover $\tilde{H}_\infty(z)=\frac{1}{2}h_\infty(|z|)|z|^2-H_\infty(z)>0$, if $z\neq 0$.