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On a class of degenerate and singular elliptic systems in bounded domains

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ABSTRACT

This paper deals with the nonexistence and multiplicity of nonnegative, nontrivial solutions to a class of degenerate and singular elliptic systems of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(h_2(x)\nabla v) = \lambda F_v(x,u,v) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^N$, $N \geqq 2$, and $h_i : \Omega \to [0,\infty)$, $h_i \in L^1_{loc}(\Omega)$, h_i $(i = 1, 2)$ are allowed to have "essential" zeroes at some points in Ω, $(F_u, F_v) = \nabla F$, and λ is a positive parameter. Our proofs rely essentially on the critical point theory tools combined with a variant of the Caffarelli–Kohn–Nirenberg inequality in [P. Caldiroli, R. Musina, On a variational degenerate elliptic problem, NoDEA Nonlinear Differential Equations Appl. 7 (2000) 189–199].

1. Introduction and main results

In this paper, we are concerned with a class of semilinear elliptic systems of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(h_2(x)\nabla u) = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqq 2)$, $(F_u, F_v) = \nabla F$ stands for the gradient of F in the variables $w = (u, v) \in \mathbb{R}^2$ and λ is a positive parameter. We point out the fact that if $h_1(x) = h_2(x) \equiv 1$, the problem was intensively studied in the last decades. We refer to some interesting works [3,8,10,17,23].

In a recent paper [6], P. Caldiroli and R. Musina have considered the Dirichlet elliptic problem of the form

$$-\operatorname{div}\big(h(x)\nabla u\big) = f(x,u) \quad \text{in } \Omega, \tag{1.2}$$

where Ω is a (bounded or unbounded) domain in $\mathbb{R}^N$ $(N \geqq 2)$, and h is a nonnegative measurable weighted function that is allowed to have "essential" zeroes at some points in Ω, i.e., the function h can have at most a finite number of zeroes in Ω. More precisely, the authors assumed that there exists an exponent $\alpha \in (0, 2]$ such that the function h decreases more slowly than $|x - z|^\alpha$ near every point $z \in h^{-1}\{0\}$. Then, they proved some interesting compact results and obtained the existence of a nontrivial solution for (1.2) in a suitable function space using the Mountain pass theorem [1]. These results were used to

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study the existence of a solution for a class of degenerate elliptic systems by N.B. Zographopoulos [22] and G. Zhang et al. [24].

In [22], N.B. Zographopoulos considered the degenerate semilinear elliptic systems of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) = \lambda\mu(x)|u|^{\gamma-1}|v|^{\delta+1} & \text{in } \Omega, \\ -\operatorname{div}(h_2(x)\nabla v) = \lambda\mu(x)|u|^{\gamma+1}|v|^{\delta-1} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the functions $h_i \in L^1_{loc}(\Omega)$ and h_i $(i = 1, 2)$ are allowed to have "essential" zeroes at some points in Ω, the function $\mu \in L^\infty(\Omega)$ and may change sign in Ω, λ is a positive parameter and the nonnegative constants γ, δ satisfy the following conditions

$$\gamma + 1 < p < 2^\star_\alpha, \qquad \delta + 1 < q < 2^\star_\beta,$$
$$\frac{\gamma+1}{p} + \frac{\delta+1}{q} = 1, \qquad \frac{\gamma+1}{2^\star_\alpha} + \frac{\delta+1}{2^\star_\beta} < 1,$$
$$2^\star_\alpha = \frac{2N}{N-2+\alpha}, \qquad 2^\star_\beta = \frac{2N}{N-2+\beta}, \quad \alpha, \beta \in (0, 2).$$

Using arguments of Mountain pass type [1], the author showed the existence of a nontrivial solution of (1.3) in the supercritical case, i.e.

$$\frac{\gamma+1}{2} + \frac{\delta+1}{2} > 1. \tag{1.4}$$

In the critical case $\gamma = \delta = 0$, the author also established the existence of a positive principal eigenvalue λ_1 for system (1.3) and some perturbations of its.

Motivated by the results in [2,6,8,18,22], G. Zhang et al. [24] obtained some existence results for (1.1) under subcritical growth conditions and the primitive $F(x, u, v)$ being intimately related to the first eigenvalue of a corresponding linear system.

In the present paper, we consider system (1.1) with the functions h_i $(i = 1, 2)$ as in [22] and [24]. Under the suitable conditions on the nonlinearities $F_u(x, u, v)$ and $F_v(x, u, v)$, using the Minimum principle (see [20, p. 4, Theorem 1.2]) and the Mountain pass theorem of A. Ambrosetti and P. Rabinowitz [1], we show that system (1.1) has at least two nonnegative, nontrivial solutions provided that λ is large enough. We also prove that the system has no nontrivial solution in case when the parameter λ is small enough. Thus, these results are completely natural extensions from [22] and [24]. Our paper is motivated by the interesting ideas introduced in [3,10,13,16]. In order to state our main results, we introduce next some hypotheses on the structure of the problem.

Throughout this paper, we assume the functions h_1 and h_2 satisfying the following conditions:

($\mathbf{H_1}$) The function $h_1 : \Omega \to [0, \infty)$ belongs to $L^1_{loc}(\Omega)$ and there exists a constant $\alpha \geqq 0$ such that

$$\liminf_{x\to z} |x - z|^{-\alpha} h_1(x) > 0 \quad \text{for all } z \in \overline{\Omega}.$$

($\mathbf{H_2}$) The function $h_2 : \Omega \to [0, \infty)$ belongs to $L^1_{loc}(\Omega)$ and there exists a constant $\beta \geqq 0$ such that

$$\liminf_{x\to z} |x - z|^{-\beta} h_2(x) > 0 \quad \text{for all } z \in \overline{\Omega}.$$

It should be observed that a model example for ($\mathbf{H_1}$) (similar to ($\mathbf{H_2}$)) is that $h_1(x) = |x|^\alpha$ (see [11,12]). The case $\alpha = 0$ covers the "isotropic" case corresponding to the Laplacian operator. In [6], the conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) were excellently used by P. Caldiroli and R. Musina. The authors proved that if a function h satisfies the conditions as in ($\mathbf{H_1}$) (similar to ($\mathbf{H_2}$)), then there exist a finite set $Z = \{z_1, z_2, \ldots, z_k\} \subset \overline{\Omega}$ and numbers $r, \delta > 0$ such that the balls $B_i = B_r(z_i)$ $(i = 1, 2, \ldots, k)$ are mutually disjoint and

$$h(x) \geqq \delta|x - z_i|^\alpha \quad \forall x \in B_i,\ i = 1, 2, \ldots, k,$$

and

$$h(x) \geqq \delta \quad \forall x \in \overline{\Omega} \Big\backslash \bigcup_{i=1}^k B_i.$$

This says the conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) implying that the elliptic operators in system (1.1) are degenerate and singular. Moreover, the sets $Z_{h_1} = \{x \in \overline{\Omega}:\ h_1(x) = 0\}$ and $Z_{h_2} = \{z \in \overline{\Omega}:\ h_2(z) = 0\}$ are finite, the potentials $h_1(x)$ and $h_2(x)$ respectively behave like $|x|^\alpha$ and $|x|^\beta$ around their degenerate points. Such problems come from the consideration of standing waves in anisotropic Schrödinger systems (see [15]). They arise in many areas of applied physics, including nuclear physics, field