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Monotonicity of zeros of Laguerre–Sobolev-type orthogonal polynomials

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ABSTRACT

Denote by $x_{n,k}^{M,N}(\alpha)$, $k=1,\dots,n$, the zeros of the Laguerre–Sobolev-type polynomials $L_n^{(\alpha,M,N)}(x)$ orthogonal with respect to the inner product

$$\langle p,q\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + Mp(0)q(0) + Np'(0)q'(0),$$

where $\alpha>-1$, $M\geqslant 0$ and $N\geqslant 0$. We prove that $x_{n,k}^{M,N}(\alpha)$ interlace with the zeros of Laguerre orthogonal polynomials $L_n^{(\alpha)}(x)$ and establish monotonicity with respect to the parameters M and N of $x_{n,k}^{M,0}(\alpha)$ and $x_{n,k}^{0,N}(\alpha)$. Moreover, we find N_0 such that $x_{n,n}^{M,N}(\alpha)<0$ for all $N>N_0$, where $x_{n,n}^{M,N}(\alpha)$ is the smallest zero of $L_n^{(\alpha,M,N)}(x)$. Further, we present monotonicity and asymptotic relations of certain functions involving $x_{n,k}^{M,0}(\alpha)$ and $x_{n,k}^{0,N}(\alpha)$.

1. Introduction and statement of results

Consider the sequence of Laguerre–Sobolev-type polynomials $\{L_n^{(\alpha,M,N)}(x)\}_{n=0}^\infty$ which are orthogonal with respect to the inner product

$$\langle p,q\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + Mp(0)q(0) + Np'(0)q'(0), \tag{1.1}$$

where $\alpha>-1$, $M\geqslant 0$ and $N\geqslant 0$. They were defined and studied first by Koekoek and Meijer [10]. Dueñas and Marcellán [7] considered the Laguerre–Sobolev-type orthogonal polynomials $\widehat{L}_n^{(\alpha,\widehat{M},\widehat{N})}(x)$ generated by the inner product

$$\langle p,q\rangle = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + \widehat{M}p(0)q(0) + \widehat{N}p'(0)q'(0),$$

where $\alpha>-1$, $\widehat{M}\geqslant 0$ and $\widehat{N}\geqslant 0$. It is clear that the sequences $\{L_n^{(\alpha,M,N)}(x)\}_{n=0}^\infty$ and $\{\widehat{L}_n^{(\alpha,\widehat{M},\widehat{N})}(x)\}_{n=0}^\infty$ coincide when $\widehat{M}=\Gamma(\alpha+1)M$ and $\widehat{N}=\Gamma(\alpha+1)N$. Hence all the results concerning the zeros of $L_n^{(\alpha,M,N)}(x)$ obtained in this paper can be rewritten in an obvious manner substituting M by $\widehat{M}/\Gamma(\alpha+1)$ and N by $\widehat{N}/\Gamma(\alpha+1)$.

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Let $L_n^{(\alpha)}(x)$, $n = 0, 1, \ldots$, be the classical Laguerre polynomial, orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx$$

and normalized by (2.4) below. In the sequel we denote by $x_{n,k}(\alpha)$ the zeros of the Laguerre polynomial $L_n^{(\alpha)}(x)$ and by $x_{n,k}^{M,N}(\alpha)$, $x_{n,k}^{M}(\alpha)$, and $x_{n,k}^{N}(\alpha)$ the zeros of $L_n^{(\alpha,M,N)}(x)$, $L_n^{(\alpha,M,0)}(x)$, and $L_n^{(\alpha,0,N)}(x)$, respectively, all arranged in decreasing order. We prove that the zeros $x_{n,k}^{M,N}(\alpha)$ interlace with the zeros $x_{n,k}(\alpha)$ when $M, N > 0$ and establish the monotonicity of the zeros $x_{n,k}^{M}(\alpha)$ and $x_{n,k}^{N}(\alpha)$ with respect to the parameters M and N, respectively.

Theorem 1. *The inequalities*

$$x_{n,k+1}^{M,N}(\alpha) < x_{n,k+1}(\alpha) < x_{n,k}^{M,N}(\alpha) < x_{n,k}(\alpha) \tag{1.2}$$

hold for every $n \in \mathbb{N}$, $n \geqslant 2$, and each k with $1 \leqslant k \leqslant n-1$. Moreover, for every fixed n the smallest zero $x_{n,n}^{M,N}(\alpha)$ satisfies

$$x_{n,n}^{M,N}(\alpha) > 0, \quad \text{for } N < N_0,$$
$$x_{n,n}^{M,N}(\alpha) = 0, \quad \text{for } N = N_0,$$
$$x_{n,n}^{M,N}(\alpha) < 0, \quad \text{for } N > N_0,$$

where

$$N_0 = \frac{(\alpha+1)\Gamma(n-1)\Gamma(\alpha+4)}{\Gamma(n+\alpha+2)}. \tag{1.3}$$

It is quite interesting that N_0 does not depend on M.

In the case $N = 0$ we obtain the following statement which was already derived by Dueñas and Marcellán [6].

Corollary 1. *The inequalities*

$$0 < x_{n,k+1}^{M}(\alpha) < x_{n,k+1}(\alpha) < x_{n,k}^{M}(\alpha) < x_{n,k}(\alpha) \tag{1.4}$$

hold for every $n \in \mathbb{N}$, $n \geqslant 2$, and each k with $1 \leqslant k \leqslant n-1$. Moreover, the smallest zero $x_{n,n}^{M}(\alpha)$ behaves like $\mathcal{O}(1/M)$ as M goes to infinity.

When $M = 0$ Theorem 1 yields:

Corollary 2. *The inequalities*

$$x_{n,k+1}^{N}(\alpha) < x_{n,k+1}(\alpha) < x_{n,k}^{N}(\alpha) < x_{n,k}(\alpha) \tag{1.5}$$

hold for every $n \in \mathbb{N}$, $n \geqslant 2$, and each k with $1 \leqslant k \leqslant n-1$. Moreover, the smallest zero $x_{n,n}^{N}(\alpha)$ satisfies

$$x_{n,n}^{N}(\alpha) > 0, \quad \text{for } N < N_0,$$
$$x_{n,n}^{N}(\alpha) = 0, \quad \text{for } N = N_0,$$
$$x_{n,n}^{N}(\alpha) < 0, \quad \text{for } N > N_0,$$

where N_0 is given by (1.3).

Setting $N_0 = \widehat{N}_0/\Gamma(\alpha+1)$, we conclude that the smallest zero $\widehat{x}_{n,n}^{\widehat{N}}(\alpha)$ of the nth Laguerre–Sobolev-type orthogonal polynomial defined by Dueñas and Marcellán [7] satisfies

$$\widehat{x}_{n,n}^{\widehat{N}}(\alpha) > 0, \quad \text{for } \widehat{N} < \widehat{N}_0,$$
$$\widehat{x}_{n,n}^{\widehat{N}}(\alpha) = 0, \quad \text{for } \widehat{N} = \widehat{N}_0,$$
$$\widehat{x}_{n,n}^{\widehat{N}}(\alpha) < 0, \quad \text{for } \widehat{N} > \widehat{N}_0,$$

where