J. Math. Anal. Appl. 368 (2010) 320–329

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global existence of solutions for the heat equation with a nonlinear boundary condition

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ARTICLE INFO

Article history:
Received 1 July 2009
Available online 18 February 2010
Submitted by M. Nakao

Keywords:
Global existence
Large time behavior
Initial–boundary value problem
Nonlinear boundary condition

ABSTRACT

We consider the initial–boundary value problem for the heat equation with a nonlinear boundary condition:

$$\begin{cases} \partial_t u = \Delta u, & x \in \mathbf{R}^N_+,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbf{R}^N_+, \\ -\dfrac{\partial u}{\partial x_N} = u^p, & x \in \partial\mathbf{R}^N_+,\ t > 0, \end{cases}$$

where $N \geqslant 1$, $p > 1 + 1/N$, and $\varphi \in L^1(\mathbf{R}^N_+) \cap L^\infty(\mathbf{R}^N_+)$. We prove the existence of global solutions with a small initial data, and study the large time behavior of solutions.

1. Introduction

We consider the initial–boundary value problem for the heat equation with a nonlinear boundary condition:

$$\begin{cases} \partial_t u = \Delta u, & x \in \mathbf{R}^N_+,\ t > 0, \\ u(x,0) = \varphi(x) \geqslant 0, & x \in \mathbf{R}^N_+, \\ -\dfrac{\partial u}{\partial x_N} = u^p, & x \in \partial\mathbf{R}^N_+,\ t > 0, \end{cases} \tag{1.1}$$

where $\mathbf{R}^N_+ = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1},\ x_N > 0\}$, $\partial\mathbf{R}^N_+ = \{x_N = 0\}$, $N \geqslant 1$, $\partial_t u = \partial u/\partial t$, $p > 1 + 1/N$, and $\varphi \in L^1(\mathbf{R}^N_+) \cap L^\infty(\mathbf{R}^N_+)$. In this paper we prove the existence of global solutions of (1.1) if the initial data φ is sufficiently small, and study the large time behavior of solutions of (1.1).

The nonlinear boundary value problem such as (1.1) can be physically interpreted as a nonlinear radiation law, and has been studied by many mathematicians (see [1–3,9–11] and the references therein). Among others, Deng, Fila, and Levine [2] treated the parabolic system with the nonlinear boundary condition in $\mathbf{R}^N$ where the Neumann data are coupled with each other. If we assume that, for above parabolic system both one of the initial data and one of the exponent of the nonlinear terms are equal to the other ones, then the problem reduces to the scalar problem (1.1). In above case, they proved that,

(i) if $p \leqslant 1 + 1/N$, there exists no global positive solution of (1.1);
(ii) if $p > 1 + 1/N$ and φ is “large”, there exists no global solution of (1.1);
(iii) if $p > 1 + 1/N$ and φ is “small”, there exists a global solution u of (1.1).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.02.007

(See also [3].) Furthermore they proved the existence of positive bounded functions f satisfying

$$\Delta f + \frac{1}{2}x \cdot \nabla f + \frac{1}{2(p-1)} f = 0, \qquad -\frac{\partial f}{\partial x_N} = f^p \quad \text{at } x_N = 0.$$

Then, since the function

$$\bar{u}(x,t) = (1+t)^{-1/2(p-1)} f\big((1+t)^{-\frac{1}{2}} x\big)$$

is a solution of (1.1) in $\mathbf{R}^N_+ \times [0,\infty)$ with the initial data $\varphi = f(x)$, by the comparison principle, we see that, if

$$0 \leqslant \varphi(x) \leqslant f(x) \quad \text{in } \mathbf{R}^N_+, \tag{1.2}$$

then there exists a global solution of (1.1) satisfying

$$\big\| u(t) \big\|_{L^\infty(\mathbf{R}^N_+)} \leqslant \big\| \bar{u}(t) \big\|_{L^\infty(\mathbf{R}^N_+)} \leqslant t^{-\frac{1}{2(p-1)}} \| f \|_{L^\infty(\mathbf{R}^N_+)}, \quad t > 0. \tag{1.3}$$

On the other hand, for the Cauchy problem of the semilinear heat equation,

$$\partial_t u = \Delta u + u^p \quad \text{in } \mathbf{R}^N \times (0,\infty), \qquad u(x,0) = \varphi(x) \geqslant 0 \quad \text{in } \mathbf{R}^N, \tag{1.4}$$

it is well known that, there exists a positive constant δ' such that, if

$$\|\varphi\|_{L^{q_*}(\mathbf{R}^N)} < \delta' \quad \text{with } q_* = \frac{N(p-1)}{2} > 1, \tag{1.5}$$

then there exists a global solution u of (1.4) such that

$$\big\| u(t) \big\|_{L^q(\mathbf{R}^N)} \asymp t^{-\frac{N}{2}(1-\frac{1}{q})} \tag{1.6}$$

as $t \to \infty$ for any $q \in [1,\infty]$ (see, for example, [4] and [8]).

In this paper we prove that, if

$$\|\varphi\|_{L^1(\mathbf{R}^N_+)} \|\varphi\|^{N(p-1)-1}_{L^\infty(\mathbf{R}^N_+)} \tag{1.7}$$

is sufficiently small, then there exists a solution of (1.1) in $\mathbf{R}^N_+ \times (0,\infty)$ satisfying (1.6), and study the large time behavior of the solution of (1.1). We remark that the quantity (1.7) is invariant in the self-similar transformation to the problem (1.1) (see Remark 1.1).

Following [2], we introduce the following two operators $S(t)$ and $S_N(t)$, and give the definition of the solution of (1.1). For any function $\omega(x', x_N) \in L^q(\mathbf{R}^N_+)$ $(q \in [1,\infty])$, we define

$$\big[S(t)\omega\big](x', \cdot) = \int_{\mathbf{R}^{N-1}} (4\pi t)^{-\frac{N-1}{2}} \exp\left(-\frac{|x'-y'|^2}{4t}\right) \omega(y', \cdot)\, dy', \tag{1.8}$$

$$\big[S_N(t)\omega\big](\cdot, x_N) = \int_0^\infty (4\pi t)^{-\frac{1}{2}} \left(\exp\left(-\frac{(x_N - y_N)^2}{4t}\right) + \exp\left(-\frac{(x_N + y_N)^2}{4t}\right)\right) \omega(\cdot, y_N)\, dy_N. \tag{1.9}$$

Let $0 < \tau < \infty$ and $\varphi \in L^\infty(\mathbf{R}^N_+)$. Then we say that u is a solution of (1.1) in $\mathbf{R}^N_+ \times (0,\tau)$ if, for any $\sigma \in (0,\tau)$, $u \in L^\infty(0,\sigma; L^\infty(\mathbf{R}^N_+))$ and u satisfies

$$u(x', x_N, t) = \big[S(t)S_N(t)\varphi\big](x', x_N) + K(x', x_N, t) \tag{1.10}$$

for any $(x', x_N, t) \in \mathbf{R}^{N-1} \times \mathbf{R}_+ \times (0,\sigma)$. Here

$$K(x', x_N, t) = \int_0^t \big(\pi(t-\eta)\big)^{-\frac{1}{2}} \exp\left(-\frac{x_N^2}{4(t-\eta)}\right) \big[S(t-\eta)u^p(\eta)\big](x', 0)\, d\eta. \tag{1.11}$$

Then we see that u is a unique classical solution of (1.1) (see also [2] and [3]). Furthermore we put

$$T_{\max} = \sup\big\{\tau \in (0,\infty)\colon u \text{ is a solution of (1.1) in } \mathbf{R}^N_+ \times (0,\tau)\big\}.$$

If $T_{\max} < \infty$, then $\limsup_{t \to T_{\max}} \|u(t)\|_{L^\infty(\mathbf{R}^N_+)} = \infty$ (see Lemmas 2.1 and 2.2), and we call $T_{\max}$ the blow-up time of the solution u.

Now we are ready to state the main result of this paper. The result gives a sufficient condition for the global existence of the solution u of (1.1), which behaves like the one of the heat equation in $\mathbf{R}^N$. In what follows we write $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbf{R}^N_+)}$ for simplicity.