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The Cauchy problem for the elliptic–hyperbolic Davey–Stewartson system in Sobolev space

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ABSTRACT

We study the initial value problem for the elliptic–hyperbolic Davey–Stewartson systems

$$\begin{cases} i\partial_t u + \Delta u = c_1|u|^2u + c_2 u\partial_{x_1}\varphi, \quad (t,x) \in \mathbb{R}^3, \\ (\partial_{x_1}^2 - \partial_{x_2}^2)\varphi = \partial_{x_1}|u|^2, \\ u(0,x) = \phi(x), \end{cases} \tag{0.1}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, $c_1, c_2 \in \mathbb{R}$, u is a complex valued function and φ is a real valued function. Our purpose is to prove the local existence and uniqueness of the solution for (0.1) in the Sobolev space $H^{3/2+}(\mathbb{R}^2)$ with small mass. Our methods rely heavily on Hayashi and Hirata (1996) [11], but we improve partial results of it, which got global existence of small solutions to (0.1) in weighted Sobolev space $H^{3,0} \cap H^{0,3}$. Our main new tools are Kenig–Ponce–Vega type commutator estimate in Kenig, Ponce and Vega (1993) [16] and its variant form.

1. Introduction

We consider the initial value problem for the Davey–Stewartson (DS) systems

$$\begin{cases} i\partial_t u + c_0\partial_{x_1}^2 + \partial_{x_2}^2 u = c_1|u|^2u + c_2 u\partial_{x_1}\varphi, \quad (t,x) \in \mathbb{R}^3, \\ (\partial_{x_1}^2 + c_3\partial_{x_2}^2)\varphi = \partial_{x_1}|u|^2, \quad u = u(t,x),\ \varphi = \varphi(t,x), \\ u(0,x) = \phi(x), \end{cases} \tag{1.1}$$

where $c_0, c_3 \in \mathbb{R}$, $c_1, c_2 \in \mathbb{C}$, u is a complex valued function and φ is a real valued function. The system (1.1) was derived by Davey and Stewartson [5] and models the evolution of two-dimensional long waves over finite depth liquid. When the capillary effects are significant, Djordjevic and Redekopp [6] and Benney and Roskes [2] showed that parameter c_3 can be negative. Ablowitz and Haberman [1] and Cornille [3] obtained a particular form of (1.1) as an example of a completely integrable model which generalizes the one-dimensional nonlinear Schrödinger equation with a cubic nonlinearity. In the inverse scattering literature, $(c_0, c_1, c_2, c_3) = (1, -1, 2, -1)$, $(-1, -2, 1, 1)$ or $(-1, 2, -1, 1)$ the system (1.1) is known as DSI, defocusing DSII and focusing DSII respectively. Ghidaglia and Saut [7] classified (1.1) as elliptic–elliptic, elliptic–hyperbolic, hyperbolic–elliptic and hyperbolic–hyperbolic according to respective signs of (c_0, c_3): $(+,+)$, $(+,-)$, $(-,+)$, $(-,-)$. The elliptic–elliptic and hyperbolic–elliptic cases are simpler because the main nonlinear term is $(\Delta^{-1}\partial_{x_1}^2|u|^2)u$, which is very similar to the cubic Schrödinger equation, we refer to [8] and reference therein for further results.

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In this paper we mainly consider the elliptic–hyperbolic case $(c_0, c_3) = (1, -1)$, furthermore, we assume that φ satisfies the radiation condition:

$$\lim_{x_2\to\infty} \varphi(t, x_1, x_2) = \varphi_1(t, x_1), \qquad \lim_{x_1\to\infty} \varphi(t, x_1, x_2) = \varphi_2(t, x_2),$$

then after a rotation in the (x_1, x_2)-plane, (1.1) can be written as

$$\begin{cases} (i\partial_t + \Delta)u = \left(c_1 + \dfrac{c_2}{2}\right)|u|^2u - \dfrac{c_2}{4}\left(\displaystyle\int_{x_1}^{\infty} \partial_{x_2}|u|^2\,dx_1' + \int_{x_2}^{\infty} \partial_{x_1}|u|^2\,dx_1'\right)u + \dfrac{c_2}{\sqrt{2}}(\partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2)u, \\ u(0, x) = \phi(x). \end{cases} \tag{1.2}$$

The well-posedness on (1.2) has attracted many attentions, before stating the previous works, we define the weighted Sobolev space as follows:

$$H^{m,l} = \left\{f \in L^2;\ \|f\|_{H^{m,l}} = \left\|(1-\Delta)^{m/2}\left(1+|x|^2\right)^{l/2} f\right\|_{L^2}\right\},$$
$$H^{m,l}_{x_j} = \left\{f \in L^2_{x_j};\ \|f\|_{H^{m,l}_{x_j}} = \left\|\left(1-\partial^2_{x_j}\right)^{m/2}\left(1+x_j^2\right)^{l/2} f\right\|_{L^2_{x_j}}\right\},$$

where $m, l \geqslant 0$. For simplicity we write $L^p_{x_1}L^q_{x_2} = L^p_{x_1}(\mathbb{R}; L^q_{x_2}(\mathbb{R}))$, $\|\cdot\| = \|\cdot\|_{L^2}$, $H^m = H^{m,0}$, $H^m_{x_j} = H^{m,0}_{x_j}$, $D_{x_j} = -i\partial_{x_j}$, and $\langle D_{x_j}\rangle = (1-\partial^2_{x_j})$. Local existence of small solutions to (1.2) was proved in [17,9] when the initial data belongs to $H^{m,0} \cap H^{0,l}$ $(m, l, > 1)$. Ref. [3] showed local existence for small data in H^m for sufficient large m, which was improved by [12] to $H^{5/2}$. Global existence of small solutions to (1.2) was shown in [14] when the data are real analytic and satisfy some decay condition. Global theory for (1.2) was also studied by [3,11,13] in usual weighted Sobolev spaces.

In this paper we only consider the L^2-conservation case of (1.2). Especially, we only consider the following case:

$$\begin{cases} (i\partial_t + \Delta)u = d_1|u|^2u + d_2\left(\displaystyle\int_{x_1}^{\infty} \partial_{x_2}|u|^2\,dx_1' + \int_{x_2}^{\infty} \partial_{x_1}|u|^2\,dx_1'\right)u + d_3(\Phi_1 + \Phi_2)u, \\ u(0, x) = \phi(x), \end{cases} \tag{1.3}$$

where $d_1, d_2, d_3 \in \mathbb{R}$ and $\Phi_1 = \Phi_1(t, x_1)$, $\Phi_2 = \Phi_1(t, x_2)$ are real valued. It is easy to verify that (1.3) obeys L^2 Conservation Law. For system (1.3), the existence of unique solution in H^s, for $s \gg 1$ was already known (cf. [3,11,12]).

When we apply the classical energy method, the main difficulty arise from the nonlocal terms, which contain derivatives. Direct calculation gives

$$\begin{aligned} \left|\left(u\int_{x_1}^{\infty} \partial_{x_2}|u|^2\,dx_1', u\right)\right| &\leqslant \left\|\int_{x_1}^{\infty} D^{1/2}_{x_2}|u|^2\,dx_1'\right\|_{L^2_{x_2}L^\infty_{x_1}} \left\|D^{1/2}_{x_2}|u|^2\right\|_{L^2_{x_2}L^1_{x_1}} \\ &\leqslant \left\|D^{1/2}_{x_2}|u|^2\right\|_{L^2_{x_2}L^1_{x_1}} \left\|D^{1/2}_{x_2}|u|^2\right\|_{L^2_{x_2}L^1_{x_1}} \end{aligned} \tag{1.4}$$

where $(\cdot,\cdot)$ is the inner product in L^2, and $\widehat{D^{1/2}_{x_2} f} = |\xi_2|^{1/2}\hat{f}(\xi)$. The main task in this paper is to control such terms arise from (1.4), which contain one half derivative, so it seems no hope to get proper control by directly applying the energy method. We follow the methods developed by Hayashi and Hirata [11] here, and our main new tools are Kenig–Ponce–Vega type commutator estimate and its variant form (see Section 3), these commutator estimates help us to get better nonlinear estimates (see Section 7), which only need $3/2 + \varepsilon$ derivative regularity. Our results improve partial results in [11], and get the following

Theorem 1.1. *For any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, assume $\phi \in H^{\frac{3}{2}+\varepsilon}$, $\|\phi\|_{L^2} < \delta$, $\Phi_1 \in C(\mathbb{R}; H^{\frac{3}{2}+\varepsilon}_{x_1})$ and $\Phi_2 \in C(\mathbb{R}; H^{\frac{3}{2}+\varepsilon}_{x_2})$. Then there exist a $T > 0$ and a unique solution u of (1.3) such that*

$$u \in C\left([0, T]; H^{\frac{1}{2}+\varepsilon}\right) \cap L^\infty\left([0, T]; H^{\frac{3}{2}+\varepsilon}\right),$$
$$\left\|u(t)\right\|_{L^2} = \|\phi\|_{L^2}.$$

Remark 1.2. Here we say that u is a solution to (1.3) in Banach X, if u solves (1.3) weakly and u is a limit of a sequence of Schwartz solutions in X.

Remark 1.3. System (1.3) is mass critical, it seems that we can't expect to drop the small mass condition $\|\phi\|_{L^2} < \delta$ easily.