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A strong convergence theorem for solutions to a nonhomogeneous second order evolution equation

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ABSTRACT

In this paper, we establish the strong convergence of possible solutions to the following nonhomogeneous second order evolution system

$$\begin{cases} u''(t) + cu'(t) \in Au(t) + f(t) & \text{a.e. } t \in (0, +\infty), \\ u(0) = u_0, & \sup_{t \geqslant 0} |u(t)| < +\infty \end{cases}$$

to an element of $A^{-1}(0)$, with an exponential rate of convergence when $f \equiv 0$, where A is a general maximal monotone operator in a real Hilbert space H, $c > 0$ is a real constant and $f : \mathbb{R}^+ \to H$ is a given function. We show also that the curve u is almost nonexpansive, and present some applications of our result.

1. Introduction

Let H be a real Hilbert space with inner product $(.,.)$ and norm $|.|$. We denote strong convergence in H by $\to$, and weak convergence by $\rightharpoonup$. $u'(t)$ (resp. $u''(t)$) denotes $\frac{du}{dt}(t)$ (resp. $\frac{d^2u}{dt^2}(t)$). A (nonlinear) possibly multivalued operator in H is a nonempty subset A of $H \times H$. A is said to be monotone if $(y_2 - y_1, x_2 - x_1) \geqslant 0$ for all $[x_i, y_i] \in A$, $i = 1, 2$. A is maximal monotone if A is monotone and $R(I + A) = H$, where I is the identity operator on H. See [3,4] for more details.

Existence, as well as asymptotic behavior of solutions to second order evolution systems of the form

$$\begin{cases} u''(t) \in Au(t) & \text{a.e. on } \mathbb{R}^+, \\ u(0) = u_0, & \sup_{t \geqslant 0} |u(t)| < +\infty \end{cases}, \tag{1}$$

were studied by many authors, among them, by Barbu [3], Bruck [5], Morosanu [13,14], Mitidieri [11,12], Poffald and Reich [15,16], and the references therein. Véron [17] showed that even for $A = \partial\varphi$, solutions to (1) may not converge strongly as $t \to +\infty$, although they always converge weakly.

In this paper, we study the strong convergence of possible solutions to the following nonhomogeneous second order evolution system

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$$\begin{cases} u''(t) + cu'(t) \in Au(t) + f(t) \quad \text{a.e. on } \mathbb{R}^+, \\ u(0) = u_0, \qquad \sup_{t \geqslant 0} |u(t)| < +\infty, \end{cases} \tag{2}$$

where A is a general maximal monotone operator in H, and $c > 0$. With suitable conditions on f, we show that the solution always converges strongly to an element of $A^{-1}(0)$, actually with an exponential rate of convergence in the homogeneous case ($f(t) \equiv 0$). This study is motivated as a continuation of our previous work in [8–10], where the asymptotic behavior of solutions to (2) was investigated for $c \leqslant 0$. Only weak convergence of solutions occurs in general in this case, and nothing was known about the case $c > 0$. Amazingly, here we are able to prove the strong convergence of solutions for the case $c > 0$, therefore giving also a strongly convergent process for approximating the zeros of a maximal monotone operator. It is also worth mentioning that besides the applications of our results to partial differential equations and optimization problems mentioned in Sections 3 and 4, our results are new even for the one-dimensional case of ordinary differential equations (where of course weak and strong convergence coincide), such as e.g. bounded solutions to the following ordinary differential equation: $u''(t) + 2u'(t) = u(t)^3 - \exp(-6t)$, $u(0) = 1$.

Existence theorems for (2) were studied by Véron [18,19] for $f(t) \equiv 0$, and by Apreutesei [1,2] for appropriate functions f.

Throughout the paper we assume that f satisfies the following assumption:

$$\text{There exists } t_0 > 0 \text{ such that } \int_{t_0}^{+\infty} t\left|f(t) - f_\infty\right| dt < +\infty \tag{3}$$

(i.e. $t(f(t) - f_\infty) \in L^1((t_0, +\infty); H)$), for some $f_\infty \in H$. By replacing $f(t)$ by $f(t) - f_\infty$, and A by $A + f_\infty$, we may assume without loss of generality that $f_\infty = 0$. Now we recall and introduce some notations and definitions we shall use in what follows.

Definition 1.1. A curve u in H is a function $u \in C([0, +\infty[, H)$. We denote $\sigma_T := \frac{1}{T}\int_0^T u(t)\,dt$ for $T > 0$.

Definition 1.2. By a solution u to (2) we mean a function $u \in C([0, T]; H) \cap H^2_{loc}((0, T); H)$ for every $T > 0$, that satisfies (2) for a.e. $t \in \mathbb{R}^+$.

We note that in this case u and u' are absolutely continuous functions on each compact subinterval of $\mathbb{R}^+$.

Definition 1.3. The curve u in H is said to be almost nonexpansive if

$$\left|u(t+h) - u(s+h)\right|^2 \leqslant \left|u(t) - u(s)\right|^2 + \varepsilon(s, t), \quad \forall s, t, h \geqslant 0,$$

where $\lim_{s,t \to +\infty} \varepsilon(s, t) = 0$. See [6,7].

2. Strong convergence theorem

In this section we establish the strong convergence of possible solutions to (2) as $t \to +\infty$, to an element of $A^{-1}(0)$. First we prove the following lemmas.

Lemma 2.1. *Assume $h : \mathbb{R}^+ \to \mathbb{R}$ is bounded above and absolutely continuous on every compact subinterval. Then*

$$\liminf_{t \to +\infty} h'(t) \leqslant 0.$$

Proof. Suppose to the contrary that $\liminf_{t \to +\infty} h'(t) \geqslant \lambda > 0$. Integrating on $[t_0, t]$, we get

$$h(t) - h(t_0) \geqslant \lambda(t - t_0).$$

Letting $t \to +\infty$, we get a contradiction. □

Lemma 2.2. *Assume $h : \mathbb{R}^+ \to \mathbb{R}$ is bounded above and h and h' are absolutely continuous on every compact subinterval. If there exists t_0 such that $h''(t) \geqslant -g(t)$ for all $t \geqslant t_0$, where $g(t) \geqslant 0$ for all $t \geqslant t_0$, then*

$$h(t) \leqslant h(s) + \int_s^\infty r g(r)\,dr$$

for all $t \geqslant s \geqslant t_0$.