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# Existence, uniqueness and attractiveness of a pseudo almost automorphic solutions for some dissipative differential equations in Banach spaces

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ABSTRACT

We give sufficient conditions ensuring the existence, uniqueness and global attractiveness of a pseudo compact almost automorphic solution of the following differential equation:

$$x'(t) = f\big(t, x(t)\big)$$

in a Banach space $E$, where $f : \mathbb{R} \times E \to E$ is a pseudo almost automorphic function with respect to the first argument. We essentially assume that the function $f$ is dissipative. Then we apply the main results to the following equation:

$$x'(t) + q(t)\big\|x(t)\big\|^{\alpha} x(t) = f(t) \quad (\alpha \geqslant 0).$$



## 1. Introduction

The aim of this work is to investigate the existence, uniqueness and attractiveness of a pseudo compact almost automorphic solution for the following ordinary differential equation:

$$x'(t) = f\big(t, x(t)\big), \tag{1.1}$$

where $E$ is a Banach space and $f : \mathbb{R} \times E \to E$ is pseudo almost automorphic with respect to the first argument. We essentially assume that $f$ is dissipative in the following sense:

$$\big[x - y, f(t, x) - f(t, y)\big]_{-} \leqslant p(t)\|x - y\|^{1+\alpha} \quad (\alpha \geqslant 0), \tag{1.2}$$

where $[x, h]_{-}$ denotes the lower semi-inner product defined as the limit of the quotient $\frac{\|x\|-\|x-th\|}{t}$ when $t \to 0^{+}$ and $p$ is a smaller function than an almost periodic function with negative mean value.

For the almost periodic solutions, when $\alpha = 0$, the question of existence and uniqueness is treated by Ait Dads et al. in [2]. In the linear case, the same authors studied the existence and uniqueness of a pseudo almost periodic solution in [1]. In [8], Ezzinbi et al. established sufficient conditions for the existence and uniqueness of the almost automorphic solution when $\alpha = 0$ and $p(t) = \text{constant} < 0$. Then they applied their results to show the existence and uniqueness of an almost automorphic solution of the following functional differential equation:

$$x'(t) = F\big(t, x(t), x_t\big), \tag{1.3}$$

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where $F:\mathbb{R}\times E\times C([-r,0];E)\to E$ is a continuous function, where $C([-r,0];E)$ is the space of all continuous functions from $[-r,0]$ to $E$ equipped with the uniform norm topology; for each $t\geqslant 0$, as usual, the history function $x_t\in C([-r,0];E)$ is defined by

$$x_t(\theta):=x(t+\theta)\quad \text{for } \theta\in[-r,0].$$

The pseudo almost automorphic case is treated in [9] by the same authors.

To state our results, we assume a more general assumption than (1.2), namely:

$$\big[x-y, f(t,x)-f(t,y)\big]_-\leqslant p(t)\|x-y\|\theta\big(\|x-y\|\big), \tag{1.4}$$

where $\theta:\mathbb{R}\to\mathbb{R}$ is continuous, nondecreasing and $\theta$ satisfying $\theta(u)>0$ for $u>0$. Then we apply our result to the following differential equation:

$$x'(t)+q(t)\big\|x(t)\big\|^{\alpha}x(t)=f(t)\quad (\alpha\geqslant 0), \tag{1.5}$$

where $q$ is a pseudo almost periodic function with a positive mean value. Recently, in [3], we treated the pseudo almost periodic case and here we propose to extend this last paper to the pseudo almost automorphic case.

Eq. (1.5) was introduced by Bayliss in [5] with $q(t)=1$ for each $t\in\mathbb{R}$. Eq. (1.5) was also considered by Arino and Hanebaly [4,10,11] who extended Bayliss results from Hilbert spaces to Banach spaces, however assuming $\alpha<1$. In [4], the authors proved with $q(t)=1$, $0\leqslant\alpha<1$, and $f$ being almost periodic function that Eq. (1.5) has one and only one almost periodic solution. This property still holds for all $\alpha\geqslant 0$ in Hilbert spaces.

Recall that almost automorphic functions are more general than almost periodic functions. They were introduced by Bochner [6], for more details about this topics we refer the reader to the book [15] where an important overview is given on almost automorphic functions. A pseudo almost automorphic function is the sum of a pseudo almost automorphic function and a ergodic perturbation. These functions were introduced recently in [12] and [13], where the authors studied some fundamental properties of pseudo almost automorphic functions.

This work is organized as follows: in Section 2 we recall some notations and definitions on pseudo almost automorphic functions, we also give the list of assumptions which will be made in the whole of this work. In Section 3, we establish some fundamental prior estimations that will be the working tools to develop the main results of this work. In Section 4, we state a result of the existence, the uniqueness and the global attractiveness of the compact almost automorphic solution. In Section 5, we prove our main result on the existence, uniqueness and global attractiveness of a pseudo compact almost automorphic solution. The last section is devoted to some applications and examples.

## 2. Notations, definitions and hypotheses

Concerning notations and definitions, throughout this paper, we denote by $(E,\|\cdot\|)$ a Banach space. The lower semi-inner product is defined by

$$[x,h]_-:=\lim_{t\to 0^+}\frac{\|x\|-\|x-th\|}{t}$$

and the upper semi-inner product by

$$[x,h]_+:=\lim_{t\to 0^+}\frac{\|x+th\|-\|x\|}{t}.$$

For some preliminary results on semi-inner product, we refer to [14].

Let $BC(\mathbb{R},E)$ be the space of all bounded and continuous functions from $\mathbb{R}$ to a Banach space $E$, equipped with the uniform topology. Let $x\in BC(\mathbb{R},E)$ and $\tau\in\mathbb{R}$. We define the function $x_\tau$ by

$$x_\tau=x(\tau+s)\quad \text{for } s\in\mathbb{R}.$$

A bounded continuous function $x:\mathbb{R}\to E$ is said to be *almost periodic* if $\{x_\tau;\tau\in\mathbb{R}\}$ is relatively compact in $BC(\mathbb{R},E)$. Denote by $AP(\mathbb{R},E)$ the set of all such functions. If $x$ is almost periodic, the *mean value*

$$\mathcal{M}\big\{x(t)\big\}_t:=\lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}x(t)\,dt$$

exists, furthermore

$$\lim_{r\to+\infty}\frac{1}{2r}\int_{-r+a}^{r+a}x(t)\,dt=\mathcal{M}\big\{x(t)\big\}_t,$$

uniformly with respect to $a\in\mathbb{R}$. For some preliminary results on almost periodic functions, we refer the reader to [7].