J. Math. Anal. Appl. 354 (2009) 648–656

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On L^1-solutions of a two-direction refinement equation ☆

Janusz Morawiec

Institute of Mathematics, Silesian University, Bankowa 14, PL-40-007 Katowice, Poland

A R T I C L E I N F O

Article history:
Received 4 January 2008
Available online 27 January 2009
Submitted by D. Waterman

Keywords:
Two-direction refinement equations
L^1-solutions
Iterated function systems
Markov operators
First sum rule condition

A B S T R A C T

In this paper we examine L^1-solutions of the following two-direction refinement equation

$$f(x)=\sum_{n=-N}^{N} c_{n,1} f(kx-n)+\sum_{n=-N}^{N} c_{n,-1} f(-kx+n).$$

We prove that the vector space of all L^1-solutions of the above equation is at most one-dimensional and consists of compactly supported functions of constant sign. We also show that in many interesting cases any L^1-solution of the two-direction refinement equation is either positive or negative on its support. Next we present sufficient conditions (easy for verification) for the existence of nontrivial L^1-solutions of the two-direction refinement equation as well as for the nonexistence of such solutions.

1. Introduction

Fix integers $k \geqslant 2$, $N \geqslant 1$ and a matrix

$$\mathbf{C}=\begin{bmatrix} c_{-N,-1} & c_{-N+1,-1} & \cdots & c_{N,-1} \\ c_{-N,1} & c_{-N+1,1} & \cdots & c_{N,1} \end{bmatrix}$$

of nonnegative reals such that

$$\sum_{(n,\varepsilon)\in\mathbf{S}} c_{n,\varepsilon}=k, \tag{1.1}$$

where $\mathbf{S}=\{(n,\varepsilon)\colon\ c_{n,\varepsilon}>0\}$. In this paper we are interested in L^1-solutions $f:\mathbb{R}\to\mathbb{R}$ of the following two-direction refinement equation

$$f(x)=\sum_{(n,\varepsilon)\in\mathbf{S}} c_{n,\varepsilon} f(\varepsilon kx-n). \tag{1.2}$$

If the first row of the matrix $\mathbf{C}$ consists of zeros, then Eq. (1.2) takes the form

$$f(x)=\sum_{n=-N}^{N} c_{n,1} f(kx-n), \tag{1.3}$$

☆ This research was supported by Silesian University Mathematics Department (Iterative Functional Equations and Real Analysis program).
E-mail address: morawiec@math.us.edu.pl.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.01.041

and it is referred to as the refinement equation or the lattice two-scale difference equation or the dilation equation. Eq. (1.3) appear in different context of pure and applied mathematics (see [1] and the references therein). It find a significant application in the theory of wavelets (see [2–4]).

Nonnegative L^1-solutions of much more general equation than Eq. (1.2) arise as densities of probability distribution functions of the series

$$\sum_{n=1}^{\infty} \eta_n \prod_{k=1}^{n-1} \xi_k, \tag{1.4}$$

where $((\eta_n, \xi_n)\colon n \in \mathbb{N})$ is an independent identically distributed sequence of vectors of random variables (see [5]). If the random variable ξ_1 takes only one value k, then series (1.4) leads to Eq. (1.3) (see [6–9]). If we assume that ξ_1 takes two values k and $-k$, then series (1.4) leads to Eq. (1.2) (see [10]). It has been proved recently that Eq. (1.2) can be used for construction two-direction wavelets (see [11–14]).

This paper is organizes as follows. In the next section we show that Eq. (1.2) has at most one (up to a multiplicative constant) nontrivial L^1-solution. Next we prove that any such a solution is compactly supported with constant sign, and moreover, either positive or negative on its support, if the support is an interval. In the third section we give sufficient conditions (easy for verification) for the existence of nontrivial L^1-solutions of (1.2) as well as for the nonexistence of such solutions. The last sections includes some remarks on Eq. (1.2).

2. Basic properties of L^1-solutions of (1.2)

It is clear that the set of all L^1-solutions of (1.2) is a real vector space. Let us denote by $\mathbf{V_C}$ this space. We begin with a general result on $\mathbf{V_C}$ which is known in the case of Eq. (1.3) (see [1]).

Theorem 2.1. *Let $f \in \mathbf{V_C}$. Then*

$$\hat{f}(t) = \hat{f}(0) \lim_{l\to\infty} f_l(t)$$

for $t \in \mathbb{R}$, where

$$f_l(t) = \sum_{(n_1,\varepsilon_1,\ldots,n_l,\varepsilon_l)\in\mathbf{S}^l} \frac{c_{n_1,\varepsilon_1}\cdots c_{n_l,\varepsilon_l}}{k^l} e^{it\sum_{j=1}^{l}\frac{\varepsilon_1\cdots\varepsilon_j}{k^j}n_j}$$

and the sequence $(f_l\colon l \in \mathbb{N})$ converges uniformly on compact subsets of $\mathbb{R}$.

Proof. We first observe that the Fourier transform $\hat{f}$ satisfies

$$\hat{f}(t) = \int_{\mathbb{R}} e^{itx} f(x)\,dx = \sum_{(n,\varepsilon)\in\mathbf{S}} \frac{c_{n,\varepsilon}}{k} e^{it\frac{\varepsilon}{k}n} \hat{f}\left(\frac{\varepsilon}{k}t\right)$$

for $t \in \mathbb{R}$. Hence, by iterating,

$$\hat{f}(t) = \sum_{(n_1,\varepsilon_1,\ldots,n_l,\varepsilon_l)\in\mathbf{S}^l} \frac{c_{n_1,\varepsilon_1}\cdots c_{n_l,\varepsilon_l}}{k^l} e^{it\sum_{j=1}^{l}\frac{\varepsilon_1\cdots\varepsilon_j}{k^j}n_j} \hat{f}\left(\frac{\varepsilon_1\cdots\varepsilon_l}{k^l}t\right) \tag{2.1}$$

for $l \in \mathbb{N}$ and $t \in \mathbb{R}$.

Fix $l \in \mathbb{N}$ and $t \in \mathbb{R}$. Then

$$\begin{aligned}\left|f_l(t) - f_{l-1}(t)\right| &\leqslant \sum_{(n_1,\varepsilon_1,\ldots,n_l,\varepsilon_l)\in\mathbf{S}^l} \frac{c_{n_1,\varepsilon_1}\cdots c_{n_l,\varepsilon_l}}{k^l} \left|e^{it\frac{\varepsilon_1\cdots\varepsilon_l}{k^l}n_l} - 1\right| \\ &\leqslant \sum_{(n_1,\varepsilon_1,\ldots,n_l,\varepsilon_l)\in\mathbf{S}^l} \frac{c_{n_1,\varepsilon_1}\cdots c_{n_l,\varepsilon_l}}{k^l} \frac{N}{k^l}|t| = \frac{N}{k^l}|t|.\end{aligned}$$

In consequence $(f_l\colon l \in \mathbb{N})$ satisfies the uniform Cauchy condition, and thus converges uniformly, on every compact subset of $\mathbb{R}$. This jointly with (2.1) forces our assertion. □

For every $(n, \varepsilon) \in \mathbf{S}$ define a map $T_{n,\varepsilon}\colon \mathbb{R} \to \mathbb{R}$ by

$$T_{n,\varepsilon}(x) = \frac{x+n}{\varepsilon k}$$