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Classification of the centers, their cyclicity and isochronicity for the generalized quadratic polynomial differential systems

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ABSTRACT

In this paper we classify the centers, the cyclicity of their Hopf bifurcation and the isochronicity of the polynomial differential systems in $\mathbb{R}^2$ of degree d that in complex notation $z = x + iy$ can be written as

$$\dot{z} = (\lambda + i)z + (z\bar{z})^{\frac{d-2}{2}}\left(Az^2 + Bz\bar{z} + C\bar{z}^2\right),$$

where $d \geqslant 2$ is an arbitrary even positive integer, $\lambda \in \mathbb{R}$ and $A, B, C \in \mathbb{C}$. Note that if $d = 2$ we obtain the well-known class of quadratic polynomial differential systems which can have a center at the origin.

1. Introduction and statement of the main results

In the qualitative theory of real planar polynomial differential systems two of the main problems are the determination of limit cycles and the center-focus problem; i.e. to distinguish when a singular point is either a focus or a center. The notion of *center* goes back at least to Poincaré in [17]. He defined it for a vector field on the real plane; i.e. a singular point surrounded by a neighborhood fulfilled of closed orbits with the unique exception of the singular point.

This paper deals with the classification of the centers for a class of polynomial differential systems which generalizes the quadratic polynomial differential systems to polynomial vector fields of arbitrary degree. The classification of the centers of the polynomial differential systems started with the quadratic ones with the works of Dulac [7], Kapteyn [11,12], Bautin [2], Zoladek [20], ... see Schlomiuk [19] for an update on the quadratic centers. There are many partial results for the centers of polynomial differential systems of degree larger than 2, but we are very far to obtain a complete classification of the centers for the polynomial differential systems of degree $\geqslant 3$.

In this paper we consider the polynomial differential systems in the real (x, y)-plane that have a singular point at the origin with eigenvalues $\lambda \pm i$ and that can be written as

$$\dot{z} = (\lambda + i)z + (z\bar{z})^{\frac{d-2}{2}}\left(Az^2 + Bz\bar{z} + C\bar{z}^2\right), \tag{1}$$

where $z = x + iy$, $d \geqslant 2$ is an arbitrary even positive integer, $\lambda \in \mathbb{R}$ and $A, B, C \in \mathbb{C}$. The vector field associated to this system is formed by the linear part $(\lambda + i)z$ and by a homogeneous polynomial of degree d formed by three monomials. For such systems, we want to determine the conditions that ensure that the origin of (1) is a center. Of course these systems for $d = 2$ coincide with the class of all quadratic polynomial differential systems. So we call the class of polynomial differential systems (1) of degree $d \geqslant 2$ the *generalized quadratic systems*.

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We write

$$A = a_1 + ia_2, \qquad B = b_1 + ib_2, \quad \text{and} \quad C = c_1 + ic_2.$$

The resolution of this problem implies the effective computation of the Liapunov constants. Indeed writing (1) in polar coordinates, i.e. doing the change of variables $r^2 = z\bar{z}$ and $\theta = \arctan(\operatorname{Im} z/\operatorname{Re} z)$, system (1) becomes

$$\frac{dr}{d\theta} = \frac{\lambda r + F(\theta)r^d}{1 + G(\theta)r^{d-1}}, \tag{2}$$

where

$$\begin{aligned} F(\theta) &= (a_1 + b_1)\cos\theta - (a_2 - b_2)\sin\theta + c_1\cos 3\theta + c_2\sin 3\theta, \\ G(\theta) &= (a_2 + b_2)\cos\theta + (a_1 - b_1)\sin\theta + c_2\cos 3\theta - c_1\sin 3\theta. \end{aligned} \tag{3}$$

Note that Eq. (2) is well defined in a sufficiently small neighborhood of the origin. Therefore if system (1) has a center, then Eq. (2) defined in the plane (r, θ) when $\dot{\theta} > 0$ also has a center at the origin.

The transformation $(r, \theta) \mapsto (\rho, \theta)$ defined by

$$\rho = \frac{r^{d-1}}{1 + G(\theta)r^{d-1}} \tag{4}$$

is a diffeomorphism from the region $\dot{\theta} > 0$ into its image. As far as we know the first in use this transformation was Cherkas in [4]. If we write Eq. (2) in the variable ρ, we obtain the following Abel differential equation

$$\begin{aligned} \frac{d\rho}{d\theta} &= (d-1)G(\theta)\big[\lambda G(\theta) - F(\theta)\big]\rho^3 + \big[(d-1)\big(F(\theta) - 2\lambda G(\theta)\big) - G'(\theta)\big]\rho^2 + (d-1)\lambda\rho \\ &= U(\theta)\rho^3 + V(\theta)\rho^2 + (d-1)\lambda\rho. \end{aligned} \tag{5}$$

This kind of differential equations appeared in the studies of Abel on the theory of elliptic functions. For more details on Abel differential equations see [10], or [9].

The solution $\rho(\theta, \gamma)$ of (5) satisfying that $\rho(0, \gamma) = \gamma$ can be expanded in a convergent power series of $\gamma \geqslant 0$ sufficiently small. Thus

$$\rho(\theta, \gamma) = \rho_1(\theta)\gamma + \rho_2(\theta)\gamma^2 + \rho_3(\theta)\gamma^3 + \cdots \tag{6}$$

with $\rho_1(\theta) = 1$ and $\rho_k(0) = 0$ for $k \geqslant 2$. Let $P: [0, \gamma_0] \to \mathbb{R}$ be the Poincaré map defined by $P(\gamma) = \rho(2\pi, \gamma)$ and for a convenient $\gamma_0 > 0$. Then the values of $\rho_k(2\pi)$ for $k \geqslant 2$ control the behavior of the Poincaré map in a neighborhood of $\rho = 0$. Therefore clearly system (1) has a center at the origin if and only if $\rho_1(2\pi) = 1$ and $\rho_k(2\pi) = 0$ for every $k \geqslant 2$. Assuming that $\rho_2(2\pi) = \cdots = \rho_{m-1}(2\pi) = 0$ we say that $v_m = \rho_m(2\pi)$ is the mth *Liapunov* or *Liapunov–Abel* constant of system (1). The problem of computing the Liapunov constants for determining a center goes back to Poincaré [17] and Liapunov [13]. It is well known that the Liapunov–Abel constants v_m are homogeneous polynomials of degree $m - 1$ in the coefficients of a_i, b_i and c_i for $i = 1, 2$, see for instance [6,16].

The set of coefficients for which all the Liapunov constants vanish is called the *center variety* of the family of polynomial differential systems. By the Hilbert Basis Theorem, the center variety is an algebraic set. Now a natural question arises: *is that one can characterize the center variety of a given family of polynomial differential systems*? In other words, *can one find necessary and sufficient conditions such that a given system of the family has a center at the origin*?

The problem of studying the centers and the foci is in general very difficult since to do it requires a good knowledge, not only of the common zeros of the Liapunov constants, but also of the finite generated ideal that they generate in the ring of polynomials taking as variables the coefficients of the polynomial differential system. Furthermore in general the calculation of the Liapunov constants is not easy, and the computational complexity of finding their common zeros grows very quickly. A number of algorithms have been developed to compute them automatically up to a certain order (see [6,15,16] and the references therein). We also want to mention that even if we are able to obtain the Liapunov constants it is in general extremely difficult to decompose the resulting variety into irreducible components. If this can be done we have necessary conditions to have a center at the origin. Usually, the sufficiency conditions will follow either from proving the existence of a first integral defined in a neighborhood of the origin, or from the existence of a symmetry through the origin.

The first objective of this paper is to determine the conditions on the parameters λ, A, B and C in order that the origin of (1) be a center.

The first main result of this paper is the following.

Theorem 1. *For $d \geqslant 2$ even, system* (1) *has a center at the origin if and only if one of the following conditions holds*:

(c.1) $\lambda = 2A + \overline{B} = 0$,
(c.2) $\lambda = \operatorname{Im}(AB) = \operatorname{Im}(A^3C) = \operatorname{Im}(\overline{B}^3C) = 0$,