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Existence of solutions for $p(x)$-Laplacian equations with singular coefficients in $\mathbb{R}^N$ ☆

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ABSTRACT

In this paper, we deal with the existence of solutions for the following $p(x)$-Laplacian equations via critical point theory

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + e(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}\left(\mathbb{R}^N\right), \end{cases}$$

where $f(x,u) = \sum_{i=1}^m \lambda_i a_i(x) g_i(x,u)$, $g_i : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition, but $a_i(x)$ are singular. Especially, we obtain existence criterion for infinite many pairs of solutions for the problem, when some $a_{i_0}(x)$ can change sign and $g_{i_0}(x,\cdot)$ satisfies super-p^+ growth condition.

1. Introduction

The study of differential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [4,22,34]). The study on variable exponent problems attracts more and more interest in recent years, many results have been obtained on this kind of problems, for example [1,4–6,8–16,19–23,26–34]. On the variable exponent Sobolev spaces which have been used to study $p(x)$-Laplacian problems, we refer to [8,20,23]. On the existence of solutions for elliptic equations with variable exponent, we refer to [13,15,16,21,26–29,31–33]. In this paper, we mainly consider the existence of weak solutions for the following equation

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + e(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}\left(\mathbb{R}^N\right), \end{cases} \tag{P}$$

where $-\Delta_{p(x)} u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian; $e \in L^\infty(\mathbb{R}^N)$ and $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} e(x) = e_0 > 0$; $p \in C(\mathbb{R}^N)$ is uniformly continuous and $1 < \inf_{x\in\mathbb{R}^N} p(x) \leqslant \sup_{x\in\mathbb{R}^N} p(x) < +\infty$; f satisfies

$$f(x,t) = \sum_{i=1}^m \lambda_i a_i(x) g_i(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

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Throughout the paper, the following conditions are satisfied:

(A) For any $i = 1, \dots, m$, λ_i is a constant, $a_i \in L^{r_i(x)}(\mathbb{R}^N)$, $g_i : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition, $|g_i(x,t)| \leqslant b_i(x) + c_i|t|^{q_i(x)-1}$ for $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, where $c_i \geqslant 0$ are constants, $b_i \in L^{q_i^0 r_i^0}(\mathbb{R}^N)$, $r_i, q_i \in C(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} r_i(x) > 1$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q_i(x) \geqslant 1$, and

$$\frac{r_i(x)-1}{r_i(x)} p(x) \leqslant q_i(x) \ll \frac{r_i(x)-1}{r_i(x)} p^*(x), \quad \forall x \in \mathbb{R}^N.$$

Here $r^0(x)$ is the conjugate function of $r(x)$, namely $\frac{1}{r(x)} + \frac{1}{r^0(x)} = 1$, the notation $f_1(x) \ll f_2(x)$ means $\operatorname{ess\,inf}_{x\in\mathbb{R}^N}(f_2(x) - f_1(x)) > 0$, and

$$p^*(x) = \begin{cases} Np(x)/(N-p(x)), & p(x) < N, \\ \infty, & p(x) \geqslant N. \end{cases}$$

When $p(x) \equiv p$ (a constant), $p(x)$-Laplacian is the usual p-Laplacian. The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian (see [14]). On the p-Laplacian problems with singular coefficients, we refer to [3,7,17,18]. But the $p(x)$-Laplacian problems with singular coefficients are rare (see [15]). On the $p(x)$-Laplacian problems on unbounded domain, we refer to [16,27]. Our aim is to give several existence results of weak solutions for $p(x)$-Laplacian problems with singular coefficients on unbounded domain. Especially, we obtain existence criterion for infinite many pairs of solutions for the problem, when some $a_{i_0}(x)$ can change sign and $g_{i_0}(x,\cdot)$ satisfies super-p^+ growth condition. These results are extensions of those of p-Laplacian problems.

This paper is divided into four sections. In the second section, we introduce some basic properties of the variable exponent Sobolev spaces. In the third section, several important properties of $p(x)$-Laplacian and variational principle are presented. Finally, in the fourth section, we give several existence results of weak solutions of problem (P).

2. Weighted variable exponent Lebesgue and Sobolev spaces

In order to discuss problem (P), we need some theories on space $W^{1,p(x)}(\mathbb{R}^N)$ which are called variable exponent Sobolev spaces. Denote by $S(\mathbb{R}^N)$ the set of all measurable real functions defined on $\mathbb{R}^N$. Write

$$h^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} h(x), \qquad h^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} h(x), \quad \text{for any } h \in S(\mathbb{R}^N),$$
$$C_+(\mathbb{R}^N) = \{h \mid h \in C(\mathbb{R}^N),\ h^- \geqslant 1 \text{ for } x \in \mathbb{R}^N\},$$
$$L^{p(x)}(\mathbb{R}^N) = \left\{u \;\middle|\; u \in S(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u(x)|^{p(x)}\,dx < \infty\right\}.$$

We can introduce the norm on $L^{p(x)}(\mathbb{R}^N)$ by

$$|u|_{p(x)} = \inf\left\{\lambda > 0 \;\middle|\; \int_{\mathbb{R}^N} \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\},$$

and $(L^{p(x)}(\mathbb{R}^N), |\cdot|_{p(x)})$ becomes a Banach space, which is called variable exponent Lebesgue space.

The space $W^{1,p(x)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N t) \mid |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\},$$

and it can be endowed with the norm

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\mathbb{R}^N).$$

Proposition 2.1. *(See [8,13].)*

(i) *The space $(L^{p(x)}(\mathbb{R}^N), |\cdot|_{p(x)})$ is a separable, uniform convex Banach space, and it is conjugate space is $L^{p^0(x)}(\mathbb{R}^N)$, where $\frac{1}{p(x)} + \frac{1}{p^0(x)} = 1$. For any $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{p^0(x)}(\mathbb{R}^N)$, we have*

$$\left|\int_{\mathbb{R}^N} uv\,dx\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{(p^0)^-}\right)|u|_{p(x)}|v|_{p^0(x)}.$$

(ii) *If $\Omega \subset \mathbb{R}^N$ is open bounded, $1 \leqslant p_1,\ p_2 \in C(\Omega)$, $p_1(x) \leqslant p_2(x)$ for any $x \in \Omega$, then $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$, and the imbedding is continuous.*