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Orthogonal exponentials on the generalized three-dimensional Sierpinski gasket

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ABSTRACT

The self-affine measure $\mu_{M,D}$ corresponding to the expanding integer matrix

$$M=\begin{bmatrix} p & 0 & m\\ 0 & p & 0\\ 0 & 0 & p\end{bmatrix} \quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

is supported on the generalized three-dimensional Sierpinski gasket $T(M,D)$, where p is odd. In the present paper we show that there exist at most 7 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. This generalizes the result of Dutkay and Jorgensen [D.E. Dutkay, P.E.T. Jorgensen, Analysis of orthogonality and of orbits in affine iterated function systems, Math. Z. 256 (2007) 801–823] on the non-spectral self-affine measure problem. By using the same method, we also obtain that for self-affine measure $\mu_{M,D}$ corresponding to the expanding integer matrix

$$M=\begin{bmatrix} p & 0\\ 0 & p\end{bmatrix} \quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\},$$

where p is odd, there exist at most 5 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

1. Introduction

Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, that is, one with all eigenvalues $|\lambda_i(M)| > 1$ and $D \subseteq \mathbb{Z}^n$ be a finite subset of cardinality $|D|$. Associated with iterated function system (IFS) $\{\phi_d(x) = M^{-1}(x+d)\}_{d\in D}$, there exists a unique probability measure $\mu := \mu_{M,D}$ satisfying the self-affine identity (see [11])

$$\mu = \frac{1}{|D|}\sum_{d\in D}\mu\circ\phi_d^{-1}. \tag{1.1}$$

Such μ is supported on $T(M,D)$ and is called self-affine measure.

Recall that for a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a spectral measure if there exists a discrete set $\Lambda \subseteq \mathbb{R}^n$ such that $E_\Lambda := \{e^{2\pi i\langle\lambda,x\rangle}\colon\ \lambda\in\Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. The set Λ is then called a spectrum for μ. Spectral measure is a natural generalization of spectral set introduced by Fuglede [6] whose famous conjecture and its related problems have received much attention in the recent years (see [13,14]). The spectral self-affine measure problem at the present day consists in determining conditions under which $\mu_{M,D}$ is a spectral measure, and has been studied in the papers [8,9,12,14,16,17,21,22] (see also [23,24] for the main goal). The non-spectral self-affine measure problem originated

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from the Lebesgue measure case (see [3–7,15,19] and [1,2] where the conjecture that the disk has no more than three orthogonal exponentials is still unsolved) usually consists of the following two classes:

(I) There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$-orthogonal exponentials contain at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them (see [10,18]).
(II) There are natural infinite families of orthogonal exponentials, but none of them forms an orthogonal basis in $L^2(\mu_{M,D})$. The questions concerning this class can be found in [20].

In the present paper we will consider the questions of the class (I) for the generalized three-dimensional Sierpinski gasket. A fractal F is a set which admits a system of scale transformations; intuitively they have the property that F looks the same as the scaling is varied. Typically a fractal comes equipped with an invariant measure. However as is illustrated by such familiar cases as the Cantor set and its invariant measure, or one of the Sierpinski examples, one must pass to a limit, and the limit typically allows intricate non-linearities. A popular representation of a class of fractals is realized with a finite set of affine transformations in Euclidean space, and this is the setting for the present paper. Now classical Fourier series relies on linearity, and so asking for Fourier series in the context of fractals is a new framework. The result below indicates the limits one encounters in such an endeavor. The main theorem improves what was previously known, i.e., papers by Dutkay and Jorgensen and by Li.

We recall the following related conclusions.

(i) The plane Sierpinski gasket $T(M,D)$ corresponding to

$$M=\begin{bmatrix}2&0\\0&2\end{bmatrix}\quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},$$

Dutkay and Jorgensen [10, Theorem 5.1(ii)] proved that $\mu_{M,D}$-orthogonal exponentials contain at most 3 elements and found such 3 elements orthogonal exponentials.

(ii) The generalized plane Sierpinski gasket $T(M,D)$ corresponding to

$$M=\begin{bmatrix}2&1\\0&2\end{bmatrix}\quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},$$

see Fig. 3 and Example 3.1 in [10], by applying [3, Theorem 3.1], Dutkay and Jorgensen proved that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 7 elements.

(iii) The generalized plane Sierpinski gasket $T(M,D)$ corresponding to

$$M=\begin{bmatrix}2&p\\0&2\end{bmatrix}\quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},$$

in the recent paper, J.-L. Li [18] obtained that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements.

The three-dimensional Sierpinski gasket $T(M,D)$ corresponding to

$$M=\begin{bmatrix}p&0&0\\0&p&0\\0&0&p\end{bmatrix}\quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \tag{1.2}$$

Dutkay and Jorgensen [10, Theorem 5.1(iii)] obtained that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 256 elements. Also, the problem is proposed in J.-L. Li [18]. In the present paper, we improve their result, obtain that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 7 elements, and also generalize this result to a more general case.

2. Main result and proof

Theorem 1. *For self-affine measure $\mu_{M,D}$ corresponding to the expanding integer matrix*

$$M=\begin{bmatrix}p&0&m\\0&p&0\\0&0&p\end{bmatrix}\quad\textit{and}\quad D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \tag{2.1}$$

where p is odd, there exist at most 7 *mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.*

Proof. For the general expanding matrix $M\in M_n(\mathbb{Z})$ and finite subset $D\subseteq\mathbb{Z}^n$, the Fourier transform of the self-affine measure $\mu_{M,D}$ is

$$\hat{\mu}_{M,D}(\xi)=\int e^{2\pi i\langle\xi,t\rangle}\,d\mu_{M,D}(t)\quad\left(\xi\in\mathbb{R}^n\right).$$