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Exponential energy decay of solutions of viscoelastic wave equations

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ABSTRACT

In this paper we consider the nonlinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t = |u|^{p-1}u \quad \text{in } \Omega \times (0,\infty),$$

with initial conditions and Dirichlet boundary condition. Under some appropriate assumption on g, by introducing a potential well we establish the global existence result for the equation above. Furthermore, we investigate explicit exponential energy decay of the viscoelastic wave equation under the potential well.

1. Introduction

Consider the following wave equation with nonlinear viscoelastic term

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t|u_t|^{m-1} = f(u) & \text{in } \Omega \times (0,\infty), \\ u(t,x) = 0, & x \in \partial\Omega,\ t \geqslant 0, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$ with a smooth boundary $\partial\Omega$, $m \geqslant 0$ and g is a positive function.

Before going any further, we consider Eq. (1.1) without the viscoelastic term, $\int_0^t g(t-\tau)\Delta u(\tau)\,d\tau$. In this case, Eq. (1.1) is known as wave equation, which has been extensively studied. For Cauchy problem, Levine [10] firstly showed that the solutions with negative initial energy are non-global for some abstract wave equation with linear damping. Later Levine and Serrin [11] considered a class of the generalized abstract wave equation, and then Pucci and Serrin [16] claimed a threshold result on the global existence and blow-up for the wave equation under a potential well. Georgiev and Todorova [7] studied the existence theorem for the wave equation with the nonlinear damping term. In 2001, Levine and Todorova [12] considered the Cauchy problem of wave equation with arbitrarily positive initial energy, and claimed that there is the choice of initial datum from the energy space with arbitrarily positive initial energy such that the corresponding solution of wave equations blows up in a finite time. Furthermore, Todorova and Vitillaro [18] proved that such a choice of the initial datum is infinite.

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Recently, for Eq. (1.1) with $g \equiv 0$ and strong damping, Gazzola and Squassina [6] have studied Cauchy problem and blow-up result with arbitrarily positive initial energy. Very recently, by concavity argument the first author [20] has also established the blow-up result with arbitrarily positive initial energy for Klein–Gordon equation. For this case, Todorova [17] has studied the energy decay of wave equations.

For Eq. (1.1) with $g \not\equiv 0$, Cavalcanti et al. [2] firstly studied the case of $m = 1$, $f(u) = -|u|^{p-1}u$, and a localized damping $a(x)u_t$. They obtained an exponential rate of decay with the assumption on g, that the kernel g is of exponential decay. And they [3] also studied the case of $m \geqslant 1$ and $f(u) = -|u|^{p-1}u$. This work was later improved by Cavalcanti et al. [5], and Berrimi and Messaoudi [1]. And then Messaoudi [14] obtained the global existence of solutions for viscoelastic equation with $f(u) = |u|^{p-1}u$ and $p > 1$, and the blow-up result with negative initial energy and global existence with the sufficient small initial data. Furthermore, he improved his blow-up result in [15]. More recently, the first author [21] has investigated a sufficient condition of the initial data with arbitrarily positive initial energy such that the corresponding solution of Eq. (1.1) with $m = 1$ blows up in finite time. This result improved the blow-up results in [14,15].

For Eq. (1.1), the following problem are still open now:

(i) Whether we can establish a sharp condition of the global existence and blow-up of solutions to the wave equation (1.1) with $f(u) = |u|^{p-1}u$.
(ii) Whether energy decay rate can be investigated for the wave equation (1.1) with $f(u) = |u|^{p-1}u$.

In this paper, we will answer these open problems with $m = 1$. Thus, throughout the rest of this paper, we always let $m = 1$ and $f(u) = |u|^{p-1}u$ with $p > 1$.

Now we state some assumptions on g.

Assumption 1.1. $g \in C^1([0,\infty))$ is a non-negative and non-increasing function satisfying

$$1 - \kappa := 1 - \int_0^\infty g(\tau)\,d\tau > 0.$$

In order to express our main result, we use $\|\cdot\|_p$ to denote the norm in $L^p(\Omega)$. For simplicity, we always use $\|\cdot\|$ to denote $\|\cdot\|_2$. And let $H_0(\Omega)$ be the Sobolev space with the property: If $u \in H_0(\Omega)$, then $u(x) = \nabla u(x) \equiv 0$ for every $x \in \partial\Omega$. in addition, $*$ means the convolution.

We next state the local existence theorem for Eq. (1.1), whose proof follows from the arguments in [2,7].

Theorem 1.1. *Suppose that $1 < p < \frac{n}{n-2}$ when $n \geqslant 3$ and $1 < p < \infty$ when $n = 1, 2$, and g satisfies Assumption 1.1. For $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem (1.1) has a unique local solution*

$$u \in C\big([0,T); H_0^1(\Omega)\big), \qquad u_1 \in C\big([0,T); L^2(\Omega)\big)$$

for the maximum existence time $T > 0$.

By a formal calculation we define the energy function as

$$E(t) = \frac{1}{2}\big\|u_t(t,\cdot)\big\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t,\cdot)\big\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{1.2}$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)\big\|v(t,\cdot) - v(\tau,\cdot)\big\|_2^2\,d\tau.$$

As in [14,15], we see that

$$\frac{d}{dt}E(t) = -\int_\Omega \big|u_t(t,x)\big|^2\,dx + \frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega \big|\nabla u(\tau) - \nabla u(t)\big|^2\,dx\,d\tau - \frac{1}{2}g(t)\big\|\nabla u(t,\cdot)\big\|_2^2 \leqslant 0. \tag{1.3}$$

Now we are in a position to state our main global existence and energy decay result for Eq. (1.1).

Theorem 1.2. *Under Assumption 1.1, $m = 1$, and $f(u) = |u|^{p-1}u$ with $1 < p < \frac{n}{n-2}$, if the initial data $(u_0, u_1) \in H_0^1 \times L^2(\mathbb{R})$ satisfies*