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On an extension of the Blaschke–Santaló inequality and the hyperplane conjecture

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Abstract

Let K be a symmetric convex body and K° its polar body. Call

$$\phi(K) = \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} \langle x, y\rangle^2 \, dy \, dx.$$

It is conjectured that $\phi(K)$ is maximum when K is an ellipsoid. In particular this statement implies the Blaschke–Santaló inequality and the hyperplane conjecture. We verify this conjecture when K is restricted to be a p-ball.

Keywords: Convex bodies; Asymptotic geometric analysis; Blaschke–Santaló; Hyperplane conjecture

1. Introduction and notation

A convex body $K \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. For every convex body, its polar set is defined by

$$K^\circ = \{ y \in \mathbb{R}^n \colon \langle y, x\rangle \leqslant 1 \text{ for all } x \in K \},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. Note that if $0 \in \operatorname{int} K$ then K° is a convex body.

For $p \in [1, \infty]$, let us denote by B_p^n the unit ball of the p-norm. It is:

$$B_p^n = \left\{ x \in \mathbb{R}^n \colon \sum_{i=1}^n |x_i|^p \leqslant 1 \right\}, \qquad B_\infty^n = \{ x \in \mathbb{R}^n \colon \max |x_i| \leqslant 1 \}.$$

It is well known that the polar body of B_p^n is B_q^n where q is the dual exponent of p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). Along this paper q will always denote the dual exponent of p.

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Given two symmetric convex bodies $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, for any $p \in [1, \infty]$ one defines a symmetric convex body $A \times_p B \subset \mathbb{R}^{n+m}$ which is the unit ball of the norm given by

$$\|(x_1, x_2)\|^p_{A \times_p B} = \|x_1\|^p_A + \|x_2\|^p_B, \qquad \|(x_1, x_2)\|_{A \times_\infty B} = \max\{\|x_1\|_A, \|x_2\|_B\}.$$

Note that the polar body of $A \times_p B$ is $A^\circ \times_q B^\circ$ and $B^n_p = B^{n-1}_p \times_p [-1, 1]$.

A convex body K is said to be in isotropic position if it has volume 1 and satisfies the following two conditions:

- $\int_K x\,dx = 0$ (center of mass at 0),
- $\int_K \langle x, \theta\rangle^2\,dx = L^2_K\ \forall\theta \in S^{n-1}$,

where L_K is a constant independent of θ, which is called the isotropy constant of K. It is known that for every convex body K there exists an affine map T such that TK is isotropic. Furthermore, if both K and TK are in isotropic position, then T is an orthogonal transformation. Hence we can define the isotropy constant for every convex body and it is verified that

$$nL^2_K = \min\left\{\frac{1}{|TK|^{1+\frac{2}{n}}} \int\limits_{a+TK} |x|^2\,dx;\ a \in \mathbb{R}^n,\ T \in GL(n)\right\}.$$

This means that the isotropy position is the one which minimizes the quantity in brackets. In particular for every convex body

$$n|K|^{1+\frac{2}{n}} L^2_K \leqslant \int\limits_K |x|^2\,dx.$$

It is conjectured that there exists an absolute constant C such that for every isotropic convex body $L_K < C$. This conjecture is known as the hyperplane conjecture and can be reformulated in several equivalent ways.

We will use the notation $\widetilde{K}$ for $|K|^{-\frac{1}{n}}K$.

Given a centrally symmetric convex body K, we call

$$\phi(K) = \frac{1}{|K||K^\circ|} \int\limits_K \int\limits_{K^\circ} \langle x, y\rangle^2\,dy\,dx.$$

Note that ϕ is an affine invariant, i.e. $\phi(K) = \phi(TK)$ for all $T \in GL(n)$. It is conjectured in [6] that $\phi(K)$ is maximized by ellipsoids. It is, for every symmetric convex body $K \subset \mathbb{R}^n$

$$\phi(K) \leqslant \phi(B^n_2) = \frac{n}{(n+2)^2}. \tag{1}$$

Remark. We can also define the functional ϕ when K is not symmetric. When K is a simplex with its center of mass at the origin, it is easy to compute that $\phi(K) = \phi(B^n_2)$. We will write these computations in Appendix A.

The Blaschke–Santaló inequality [8] says that for every symmetric convex body K

$$|K||K^\circ| \leqslant |B^n_2|^2.$$

In Section 2 we will see that the conjecture (1) implies Blaschke–Santaló inequality and the hyperplane conjecture. In Section 3 we are going to prove that the conjecture is true if we restrict K to be a p-ball, for some $p \geqslant 1$. We state this as a theorem:

Theorem 1.1. *Among the p-balls, the functional ϕ is maximized for the Euclidean ball*

$$\max_{p\in[1,\infty]} \phi(B^n_p) = \phi(B^n_2) = \frac{n}{(n+2)^2}.$$