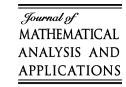




J. Math. Anal. Appl. 344 (2008) 292-300



www.elsevier.com/locate/jmaa

On an extension of the Blaschke–Santaló inequality and the hyperplane conjecture

David Alonso-Gutiérrez 1

Departamento de Matemáticas, Universidad de Zaragoza, Pedro Cerbuna 12, Zaragoza, Spain Received 31 October 2007 Available online 7 March 2008 Submitted by M. Ledoux

Abstract

Let K be a symmetric convex body and K° its polar body. Call

$$\phi(K) = \frac{1}{|K||K^{\circ}|} \int_{K} \int_{K^{\circ}} \langle x, y \rangle^{2} \, dy \, dx.$$

It is conjectured that $\phi(K)$ is maximum when K is an ellipsoid. In particular this statement implies the Blaschke–Santaló inequality and the hyperplane conjecture. We verify this conjecture when K is restricted to be a p-ball. © 2008 Elsevier Inc. All rights reserved.

Keywords: Convex bodies; Asymptotic geometric analysis; Blaschke-Santaló; Hyperplane conjecture

1. Introduction and notation

A convex body $K \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. For every convex body, its polar set is defined by

$$K^{\circ} = \{ y \in \mathbb{R}^n : \langle y, x \rangle \le 1 \text{ for all } x \in K \},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^n . Note that if $0 \in \text{int } K$ then K° is a convex body. For $p \in [1, \infty]$, let us denote by B_p^n the unit ball of the p-norm. It is:

$$B_p^n = \left\{ x \in \mathbb{R}^n \colon \sum_{i=1}^n |x_i|^p \leqslant 1 \right\}, \qquad B_\infty^n = \left\{ x \in \mathbb{R}^n \colon \max |x_i| \leqslant 1 \right\}.$$

It is well known that the polar body of B_p^n is B_q^n where q is the dual exponent of p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). Along this paper q will always denote the dual exponent of p.

E-mail address: daalonso@unizar.es.

Supported by FPU Scholarship from MEC (Spain), MCYT Grants (Spain) MTM2007-61446, DGA E-64 and by Marie Curie RTN CT-2004-511953.

Given two symmetric convex bodies $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, for any $p \in [1, \infty]$ one defines a symmetric convex body $A \times_n B \subset \mathbb{R}^{n+m}$ which is the unit ball of the norm given by

$$\|(x_1, x_2)\|_{A \times_p B}^p = \|x_1\|_A^p + \|x_2\|_B^p, \qquad \|(x_1, x_2)\|_{A \times_\infty B} = \max\{\|x_1\|_A, \|x_2\|_B\}.$$

Note that the polar body of $A \times_p B$ is $A^{\circ} \times_q B^{\circ}$ and $B_p^n = B_p^{n-1} \times_p [-1, 1]$. A convex body K is said to be in isotropic position if it has volume 1 and satisfies the following two conditions:

- $\int_K x \, dx = 0$ (center of mass at 0), $\int_K \langle x, \theta \rangle^2 \, dx = L_K^2 \ \forall \theta \in S^{n-1}$,

where L_K is a constant independent of θ , which is called the isotropy constant of K. It is known that for every convex body K there exists an affine map T such that TK is isotropic. Furthermore, if both K and TK are in isotropic position, then T is an orthogonal transformation. Hence we can define the isotropy constant for every convex body and it is verified that

$$nL_K^2 = \min \left\{ \frac{1}{|TK|^{1+\frac{2}{n}}} \int_{a+TK} |x|^2 dx; \ a \in \mathbb{R}^n, \ T \in GL(n) \right\}.$$

This means that the isotropy position is the one which minimizes the quantity in brackets. In particular for every convex body

$$n|K|^{1+\frac{2}{n}}L_K^2 \leqslant \int_K |x|^2 dx.$$

It is conjectured that there exists an absolute constant C such that for every isotropic convex body $L_K < C$. This conjecture is known as the hyperplane conjecture and can be reformulated in several equivalent ways.

We will use the notation \widetilde{K} for $|K|^{-\frac{1}{n}}K$.

Given a centrally symmetric convex body K, we call

$$\phi(K) = \frac{1}{|K||K^{\circ}|} \int_{K} \int_{K^{\circ}} \langle x, y \rangle^{2} \, dy \, dx.$$

Note that ϕ is an affine invariant, i.e. $\phi(K) = \phi(TK)$ for all $T \in GL(n)$. It is conjectured in [6] that $\phi(K)$ is maximized by ellipsoids. It is, for every symmetric convex body $K \subset \mathbb{R}^n$

$$\phi(K) \leqslant \phi(B_2^n) = \frac{n}{(n+2)^2}.$$

Remark. We can also define the functional ϕ when K is not symmetric. When K is a simplex with its center of mass at the origin, it is easy to compute that $\phi(K) = \phi(B_1^n)$. We will write these computations in Appendix A.

The Blaschke–Santaló inequality [8] says that for every symmetric convex body K

$$|K||K^{\circ}| \leqslant |B_2^n|^2$$
.

In Section 2 we will see that the conjecture (1) implies Blaschke–Santaló inequality and the hyperplane conjecture. In Section 3 we are going to prove that the conjecture is true if we restrict K to be a p-ball, for some $p \ge 1$. We state this as a theorem:

Theorem 1.1. Among the p-balls, the functional ϕ is maximized for the Euclidean ball

$$\max_{p\in[1,\infty]}\phi(B_p^n)=\phi(B_2^n)=\frac{n}{(n+2)^2}.$$

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