Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 340 (2008) 1027–1032
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A remark on vorticity maximal function ☆

Zhu Xiangrong

Department of Mathematics, Zhejiang University, China

Received 25 April 2007
Available online 21 September 2007
Submitted by B. Straughan

Abstract

In this note the following inequality is proved. For any nonnegative measure $\mu \in H^{-1}(R^2)$, $x \in R^2$ and $0 < r < 1$, there holds

$$\mu\big(B(x,r)\big) \leqslant C\left(\ln\frac{1}{r}\right)^{-\frac{1}{2}} \|\mu\|_{H^{-1}} \tag{1}$$

where C is a positive constant. Using (1) an estimate for the vorticity maximal function similar to the estimate in Majda [A. Majda, Remarks on weak solutions for vortex sheets with a distinguished sign, Indiana Univ. Math. J. 42 (1993) 921–939] is established without assuming the initial vorticity having compact support. From this a more simple proof of the Delort's celebrated theorem [J.M. Delort, Existence de mappes de fourbillon en dimension deux, J. Amer. Math. Soc. 4 (1991) 553–586] is presented.

Keywords: Euler equation; Vorticity; BMO

1. Introduction

A $2D$ ideal incompressible fluid is described by the Euler equation

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u = \nabla P, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \quad x \in R^2,\ t \in (0,\infty), \tag{2}$$

where $u = (u_1(x,t), u_2(x,t))$ and $P = P(x,t)$ denote the velocity vector and the pressure of fluid at the point $(x,t) \in R^2 \times (0,\infty)$.

Definition 1. u is called a weak solution to (2) on $(0,\infty)$ with initial data u_0, if it satisfies the following conditions:

☆ Supported by PDSFC (20060400336), NSFC (10571156, 10601046) and ZJNSF (RC97017).
E-mail address: zxr@zju.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.09.024

$$\begin{cases} \text{(a)} & u \in L^2_{\mathrm{loc}}(R^2 \times (0,\infty)); \\ \text{(b)} & \int u(t) \cdot \nabla\phi = 0 \text{ for all } \phi \in C_0^\infty(R^2),\ t \in (0,\infty); \\ \text{(c)} & \int_0^\infty \int_{R^2} (u \cdot \frac{\partial f}{\partial t} + \sum_{k,l=1}^2 u_k u_l \frac{\partial f_k}{\partial x_l})\, dx\, dt = -\int_{R^2} u_0 \cdot f(\cdot, 0) \\ & \text{for all } f \in C_0^\infty([0,\infty) \times R^2) \text{ with } \nabla \cdot f = 0. \end{cases}$$

For the smooth initial velocity with finite energy it is well known that there exists a unique global solution to (2). But for the smooth initial velocity without finite energy, the corresponding results were not presented until the last decade, cf. Serfati [15] for $C^{1+\epsilon}$, Vishik [18] for $B_{p1}^{1+2/p}$ $(p < \infty)$ and Zhu [24] for $B_{\infty 1}^1$.

If the initial vorticity is in L^p $(1 < p < \infty)$ or Zygmund class $L \ln L$, many mathematicians have done much work to obtain the global existence of the weak solution, such as [1,5,8]. More details can be found in Lions [9].

On the other hand, for the nondecaying initial vorticity L^∞ or a little larger space, Yudovich [21,22], Serfati [14] and Vishik [19,20] have obtained the uniqueness and global existence of the weak solution. Besides, in Taniuchi [16] and Zhu [23] the global existence of the weak solution has been proved for initial vorticity in *bmo* and *BMO*.

But when we concern the existence of weak solution to (2) with vortex sheet initial data, some new phenomena involving concentrations occur. With some simple examples Diperna and Majda pointed out that in a sequence of approximate solutions there may be no any subsequence converging in $L^2_{\mathrm{Loc}}(R^2)$. Therefore it is not available to use the strong convergence to obtain the weak solution as before.

Definition 2. A $2D$ incompressible velocity field $u_0 = (u_0^1(x,t), u_0^2(x,t))$ defines vortex sheet initial data provided that u_0 has locally kinetic energy, i.e.

$$\int\limits_{|x|<R} u_0^2\, dx < \infty \quad \text{for any } R > 0 \tag{3}$$

and the initial vorticity $\omega_0 = \mathrm{curl}\, u_0$ is a finite Random measure, i.e. $\omega_0 \in M(R^2)$.

Vortex sheet problem is of fundamental importance to applied mathematics and pure mathematics and many mathematicians have investigated this problem for the last two decades. Classical linearized stability analysis of the simplest vortex sheets reveals that the evolution of vortex sheets involves a classical Hadamard ill-posed initial value problem. In the series of papers of Diperna and Majda [4–6] they set up a completely new approach to study the weak solutions or measured-value solutions to the $2D$ incompressible Euler equations.

In [3] Delort made an important contribution by proving the global existence of weak solution to the $2D$ incompressible Euler system for the initial vortex sheet with a nonnegative vorticity. The main steps would be presented in the remainder of this paper. In Evans and Muller [7] and Majda [12] they gave two different proofs of Delort's theorem. In [17] Vecchi and Wu proved this theorem for the negative part of initial vorticity in $L^1 \cap H^{-1}$. In [10,11,13] they developed these ideas and got some new conclusions. But it is remain unknown now whether there exists a global weak solution for the vortex sheet initial data. More details about the vortex sheet problem can be found in Schochet [13].

Next we review the main steps in the proof of Delort for the nonnegative initial vorticity.

At first, we construct a sequence of smooth initial velocity $u_0^n \in C_0^\infty(R^2)$ such that

$$\lim_{n\to\infty} \int u_0^n f = \int u_0 f, \quad \forall f \in C_0^\infty\big(R^2\big),$$

$$\int\limits_{|x|<R} \big(u_0^n\big)^2(x)\, dx < C_R, \quad \forall R > 0,$$

$$\int \omega_0^n \leqslant C, \quad \omega_0^n \geqslant 0.$$

Then for any $n \in N$ there exists a smooth solution u^n to the Euler system with initial velocity u_0^n. If we can choose a subsequence of $\{u^n\}$ (for convenience in exposition we use the same notation) and a function u such that $\lim_{n\to\infty} \iint u_i^n u_j^n \varphi = \iint u_i u_j \varphi$ for any test function φ and $i, j = 1, 2$, then u is a weak solution of the Euler system.