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Multiplicity of positive solution for an indefinite elliptic equation on $\mathbb{R}^2$ with exponential nonlinearities ☆

Linfeng Mei

Department of Mathematics, Shandong Jianzhu University, Jinan, Shandong 250014, PR China

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Abstract

In this paper, we consider the multiplicity of positive solution to the equation

$$-\Delta u = \lambda u + h(x)u^p e^u, \quad x \in \mathbb{R}^2,$$

with $h(x)$ a sign-changing function, $p > 1$ a constant and λ a parameter. We first use a moving plane argument to get a priori bounds for the positive solutions of this equation. Then we obtain multiple positive solutions through a squeezing method, which overcomes the lack of compactness of the problem.

Keywords: Indefinite elliptic equation; Exponential nonlinearity; Multiplicity results; A priori bounds

1. Introduction

In this paper, we consider the multiplicity of positive solution to the equation

$$-\Delta u = \lambda u + h(x)u^p e^u, \quad x \in \mathbb{R}^2, \tag{1.1}$$

with $h(x)$ a sign-changing function, $p > 1$ a constant and λ is a parameter. Problem of this kind arises from a variety of situations such as prescribing curvature equation in Riemann geometry (see e.g. [7,8]) and models in population genetics (see e.g. [6]).

For presentation simplicity, we assume $h(x)$ to be sufficiently smooth. Denote

$$\Omega^+ = \{x \in \mathbb{R}^2 \mid h(x) > 0\}, \qquad \Omega^- = \{x \in \mathbb{R}^2 \mid h(x) < 0\}$$

and

$$\Gamma = \Omega^0 = \{x \in \mathbb{R}^2 \mid h(x) = 0\}.$$

We assume Ω^0 is a smooth manifold of dimension 1 and Ω^+ is a bounded set with positive measure.

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E-mail address: sdaimei@hotmail.com.

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This paper is mainly motivated by paper [10], in which Y. Du studied the problem

$$-\Delta u = \lambda u + h(x)u^p, \quad x \in \mathbb{R}^N, \tag{1.2}$$

with $1 < p < \min\{(N+1+\gamma)/(N-1), p^*\}$ ($p^* = \infty$ if $N = 1, 2$; $p^* = (N+2)/(N-2)$ if $N > 2$) and $h(x)$ sign-changing. Here γ is a positive number. He proved, provided that

$$\lim_{|x|\to\infty} h(x) = -\alpha_\infty < 0, \tag{1.3}$$

and $h(x)$ has nonzero derivative (possibly high-order) on Γ, there exists a number $\Lambda > 0$ such that (1.1) has at least two positive solutions for each $\lambda \in (0, \Lambda)$, at least one positive solution for each $\lambda \in (-\infty, 0] \cup \{\Lambda\}$ and has no positive solution when $\lambda \in (\Lambda, \infty)$.

Another paper that also inspires us is [1], in which Adimurthi and J. Giacomoni studied the problem

$$\begin{cases} -\Delta u = \lambda u + h(x)\phi(u)e^u, & x \in \mathbb{R}^2, \\ u \geqslant 0, \quad u \to 0 & \text{when } |x| \to \infty, \end{cases} \tag{1.4}$$

with $h(x)$ sign-changing, $\phi(u) \sim u^p$ ($p > 1$) near $u = 0$ and $\phi'(u)$ bounded when u is large. They first got a priori bounds for the positive solutions to Eq. (1.4) provided the derivatives of $h(x)$ does not vanish on the zero set of $h(x)$. Then they proved the existence of at least one positive solution to (1.4) for suitable λ. The a priori bounds was achieved by a moving plane method. While the existence of positive solution was obtained through globe bifurcation theory.

Although our paper is an analogue of paper [10], it includes some new ingredients.

Firstly, the adaptation of methods from paper [10] to our paper is not so trivial as it seems. Secondly, the blow-up methods used by [10] to achieve a priori bounds for the positive solutions to (1.2) can no longer be used to (1.1). Instead, we will use a moving plane process to obtain such a priori bounds. As a result, we need less restrictive conditions to obtain the a priori bounds. Yet the result we obtain is stronger than that in [10]. Using our method, we can prove the restriction $p < (N+1+\gamma)/(N-1)$ is not necessary. On the other hand, as will be seen, our assumptions required to carried on the moving plane process is much weaker than in [1].

Moreover our restrictions on $h(x)$ at infinity is different from those in [1], where $h(x)$ was required to be zero at the infinity. As will be seen, such difference causes essentially different methods in dealing with (1.1) and (1.4). We would also like to point out that in paper [1], in order to obtain the uniform boundedness of positive solutions to (1.4) over compact subsets of Ω^+, the authors used Theorem 3 in [4]. It seems that the same method cannot be used in our case without suitable modification. In this paper, we develop a lemma (see Lemma 2.3) along the argument in [4] to treat this issue.

Thirdly, paper [10] began its discuss from an established result (see, e.g., [2,14]):

When (1.2) *is considered on a bounded domain $\Omega \subset \mathbb{R}^N$ with standard boundary conditions on $\partial\Omega$, then under suitable conditions on p and on the behavior of $h(x)$ near its zero set,* (1.2) *has a positive solution for $\lambda = \lambda_1(\Omega)$* (*the first eigenvalue of the Laplacian under the corresponding boundary conditions on $\partial\Omega$*) *if and only if*

$$\int_\Omega h(x)\varphi_\Omega^{p+1}(x)\,dx < 0, \tag{1.5}$$

where φ_Ω denotes the (*normalized*) *positive eigenfunction corresponding to $\lambda_1(\Omega)$. Moreover, when* (1.5) *is satisfied, there exists $\Lambda > 0$ such that* (1.2) *has at least two positive solutions for each $\lambda \in (\lambda_1(\Omega), \Lambda)$, at least one positive solution for $\lambda = \lambda_1(\Omega)$ and for $\lambda = \Lambda$, and no positive solution for $\lambda > \Lambda$. Under less restrictive conditions,* (1.2) *has at least one positive solution for each $\lambda < \lambda_1(\Omega)$.*

However, similar result for (1.1) does not seem to have yet been established in the literature.

Our problem also differs from (1.4). Apart from the difference mentioned before, we would also like to point out that paper (1.4) imposed too restrictive conditions on $\phi(u)$. In particular, it cannot allow the nonlinearities we are dealing with.