Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 340 (2008) 487–497

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On the growth of solutions of a class of higher order linear differential equations with coefficients having the same order ☆

Jin Tu *, Cai-Feng Yi

College of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330022, China

Received 5 November 2006

Available online 29 August 2007

Submitted by A.V. Isaev

Abstract

In this paper, the authors investigate the growth of solutions of a class of higher order linear differential equations

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = 0$$

when most coefficients in the above equations have the same order with each other, and obtain some results which improve previous results due to K.H. Kwon [K.H. Kwon, Nonexistence of finite order solutions of certain second order linear differential equations, Kodai Math. J. 19 (1996) 378–387] and Z.-X. Chen [Z.-X. Chen, The growth of solutions of the differential equation $f'' + e^{-z} f' + Q(z) f = 0$, Sci. China Ser. A 31 (2001) 775–784 (in Chinese); Z.-X. Chen, On the hyper order of solutions of higher order differential equations, Chinese Ann. Math. Ser. B 24 (2003) 501–508 (in Chinese); Z.-X. Chen, On the growth of solutions of a class of higher order differential equations, Acta Math. Sci. Ser. B 24 (2004) 52–60 (in Chinese); Z.-X. Chen, C.-C. Yang, Quantitative estimations on the zeros and growth of entire solutions of linear differential equations, Complex Var. 42 (2000) 119–133].

Keywords: Linear differential equations; Entire function; Type; Hyper order

1. Introduction and results

We shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see e.g. [11,15]). In addition, we will use the notation $\sigma(f)$ to denote the order of growth of entire function $f(z)$, $\tau(f)$ to denote the type of $f(z)$ with $\sigma(f) = \sigma$, is defined to be

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{r^\sigma}.$$

☆ This work is supported by the Science Foundation of Education Department of Jiangxi Province in China.
* Corresponding author.
E-mail addresses: tujin2008@sina.com (J. Tu), yicaifeng55@163.com (C.-F. Yi).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.08.041

We use $\sigma_2(f)$ to denote the hyper order of $f(z)$, is defined to be (see [18])

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}.$$

We use mE to denote the linear measure of a set $E \subset (0,+\infty)$ and use $m_l E$ to denote the logarithmic measure of a set $E \subset [1,+\infty)$. If $P(z)$ is a polynomial, we use the notation $\deg P$ to denote the degree of $P(z)$.

For second order linear differential equations

$$f'' + B(z)f' + A(z)f = 0, \tag{1.1}$$

many authors have investigated the growth of solutions of (1.1), where $A(z) \not\equiv 0$ and $B(z)$ are entire functions of finite order. It is well known that if either $\sigma(B) < \sigma(A)$ or $\sigma(A) < \sigma(B) \leqslant 1/2$, then every solution $f \not\equiv 0$ of (1.1) is of infinite order (see [9,13]). For the case $\sigma(A) < \sigma(B)$ and $\sigma(B) > 1/2$, many authors have studied the problem. In 2000, I. Laine and P.C. Wu proved the following result.

Theorem A. *(See [16].) Suppose that $\sigma(A) < \sigma(B) < \infty$ and that $T(r,B) \sim \log M(r,B)$ as $r \to \infty$ outside a set of finite logarithmic measure. Then every non-constant solution f of (1.1) is of infinite order.*

Thus a natural question is: what condition on $A(z)$, $B(z)$ when $\sigma(A) = \sigma(B)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? For second order linear differential equations,

$$f'' + h_1 e^{P(z)} f' + h_0 e^{Q(z)} f = 0, \tag{1.2}$$

in 1996, K.H. Kwon investigated the growth of the solutions of (1.2) for the case $\deg P = \deg Q$ and obtained the following result.

Theorem B. *(See [14].) Let $P(z) = a_n z^n + \cdots$, $Q(z) = b_n z^n + \cdots$ $(a_n b_n \neq 0)$ be non-constant polynomials, $h_1(z)$ and $h_0(z) \not\equiv 0$ be entire functions with $\sigma(h_j) < n$ $(j = 0,1)$, if $\arg a_n \neq \arg b_n$ or $a_n = c b_n$ $(0 < c < 1)$, then every solution $f \not\equiv 0$ of (1.2) has infinite order with $\sigma_2(f) \geqslant n$.*

In 2001, Z.-X. Chen investigated the problem and proved the following theorem.

Theorem C. *(See [2].) Let $A_j(z) \not\equiv 0$ $(j = 0,1)$ be entire functions with $\sigma(A_j) < 1$, a, b be complex numbers such that $ab \neq 0$ and $a = cb$ $(c > 1)$. Then every solution $f \not\equiv 0$ of the equation*

$$f'' + A_1(z)e^{az} f' + A_0(z)e^{bz} f = 0 \tag{1.3}$$

has infinite order.

Combining Theorems B and C, we obtain that if $ab \neq 0$ and $a \neq b$, then every solution $f \not\equiv 0$ of (1.3) has infinite order. Can we get the similar result in higher order linear differential equations which has the same form as (1.3)? The following Corollary 3 gives the affirmative answer.

For higher order linear differential equations

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = 0, \tag{1.4}$$

Z.-X. Chen obtained the following theorems.

Theorem D. *(See [6].) Let $A_j(z)$ $(j = 0,\ldots,k-1)$ be entire functions such that*

$$\max\{\sigma(A_j),\ j = 1,\ldots,k-1\} < \sigma(A_0) < +\infty.$$

Then every solution $f \not\equiv 0$ of (1.4) satisfies $\sigma_2(f) = \sigma(A_0)$.

Theorem E. *(See [3].) Suppose that a_j $(j = 0,\ldots,k-1)$ are complex numbers. There exist a_s and a_l such that $s < l$, $a_s = d_s e^{i\varphi}$, $a_l = -d_l e^{i\varphi}$, $d_s > 0$, $d_l > 0$, and for $j \neq s, l$, $a_j = d_j e^{i\varphi}$ $(d_j \geqslant 0)$ or $a_j = -d_j e^{i\varphi}$, $\max\{d_j \mid j \neq s, l\} = d < \min\{d_s, d_l\}$. If $A_j = h_j(z)e^{a_j z}$, where h_j are polynomials, $h_s h_l \not\equiv 0$, then every transcendental solution f of (1.4) satisfies $\sigma(f) = \infty$ and $\sigma_2(f) = 1$.*