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# Spectral analysis of family of singular non-self-adjoint differential operators of even order

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**Abstract**

This work examines the spectrum of a family of certain non-self-adjoint singular differential operators of even order on a whole axis. The coefficients of such operators depend on a complex spectral parameter in a polynomial manner. The scope of our work is also engaged in the construction of the resolvent and a multiple spectral expansion which is corresponding to such operators. This process is performed under the hypothesis that the coefficients of the differential expression are not infinitely small. The similar problems on a semi-axis and a whole axis were investigated in earlier papers [F.G. Maksudov, E.E. Pashayeva, About multiple expansion in terms of eigenfunctions for one-dimensional non-self-adjoint differential operator of even order on a semi-axis, in: Spectral Theory of Operators and Its Applications, vol. 3, Elm Press, Baku, 1980, pp. 34–101 (in Russian)] and [E.E. Pashayeva, About one multiple expansion in terms of solutions of differential equation on the whole axis, in: Spectral Theory of Operators and Its Applications, vol. 5, Elm Press, Baku, 1984, pp. 145–151 (in Russian)], respectively. However, in those papers, the coefficients of the differential expression were decreasing rapidly enough as $x$ was approaching to infinity.


*Keywords:* Spectral analysis; Non-self-adjoint differential operator; Spectrum; Resolvent; Multiple expansion

## 1. Introduction

This work performs the spectral analysis of the differential operator constructed by the following differential expression

$$l_\lambda(y) = L_0 y + L_1(\lambda) y \tag{1.1}$$

where

$$L_0 = \sum_{j=0}^{2n} q_j \frac{(-id)}{dx^{2n-j}}^{2n-j}, \tag{1.2}$$

$q_0 \equiv 1$, $q_1 = 0$, coefficients $q_j$, $j = 2, \dots, 2n$, are complex-valued. Moreover

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$$L_1(\lambda) = \sum_{j=2}^{2n} P_j(x,\lambda) \frac{(-id)}{dx^{2n-j}}^{2n-j},$$

$$P_j(\lambda) = \lambda^{j-1} p_{j1}(x) + \cdots + p_{jj}(x), \tag{1.3}$$

where $p_{kj}$(x), $j \leqslant k$, are complex-valued functions.

We suppose that

$$(x^2+1)^r p_{jk}(x) \in L_2(-\infty,\infty) \tag{1.4}$$

for chosen $r$, here $r$ is the highest multiplicity of the root of $p'(\mu) = 0$ and

$$p(\mu) = \mu^{2n} + q_1\mu^{2n-1} + q_2\mu^{2n-2} + \cdots + q_{2n-1}\mu + q_{2n}. \tag{1.5}$$

With the differential expression (1.1) we associate an operator $L(\lambda)$ operating on the Hilbert space $L_2(-\infty,\infty)$. We define the operator $L(\lambda)$ by the formula $L(\lambda)y = l_\lambda(y)$ on functions $y \in L_2(-\infty,\infty)$ which have derivatives $y^{(\nu)}$, $\nu = 1,\dots,2n-1$, absolutely continuous on every interval $[-b,b]$, $b > 0$ and which are such that $l_\lambda(y) \in L_2(-\infty,\infty)$. Since we do not assume that $\operatorname{Im} p_{kj}(x) \equiv 0$, the differential expression (1.1) and the operator $L(\lambda)$ are not self-adjoint.

This work examines the solutions of the equation $l_\lambda(y) + \lambda^{2n} = 0$ and their asymptotic behaviors as $x \to \pm\infty$ and $|\lambda| \to \infty$. We also investigate the spectrum, then construct the resolvent and a multiple spectral expansion which is corresponding to the operator $L(\lambda)$. This process is performed under the hypothesis that the coefficients of the differential expression are not infinitely small. As a result of our analysis the multiple expansion of arbitrary test functions was obtained. We got the expansion in terms of the sum of eigenfunctions and the integral that involves some solutions of the corresponding differential equation.

This work generalizes some previously established results. The similar problems on a semi-axis and a whole axis were investigated in earlier papers [4] and [6], respectively. However, in those papers, the coefficients of the differential expression were decreasing rapidly enough as $x$ was approaching to infinity.

R.R.D. Kemp [2] has examined a problem for similar operator where the coefficients were not infinitely small, but they did not depend on a complex parameter. He analyzed the spectrum of a corresponding operator and obtained an expansion in characteristic functions for suitably restricted class of functions. We examine a more general spectral problem. In our case the coefficients of the differential operator are not infinitely small, but depend on complex spectral parameter. We also obtain a suitable multiple expansion of arbitrary test functions.

## 2. Solutions of the equation $l_\lambda(y) + \lambda^{2n} y = f$

Consider the case when the equation

$$p(\mu) = \lambda^{2n}$$

has no the real values.

Note that for real $t$ the equation $p(t) = \lambda^{2n}$ defines a curve in the complex $\lambda$-plane. This equation will in general split the complex $\lambda$-plane up into the regions $D_m$, $m = 1,\dots,2n$.

For large $|\lambda|$ $\mu_i$, $i = 1,\dots,2n$, can be renumbered so that

$$\mu_j = \alpha_j \lambda\bigl(1 + O\bigl(|\lambda|^{-1}\bigr)\bigr), \quad 0 \leqslant \arg\lambda \leqslant \frac{\pi}{n},\ m = 1,2,\dots,2n, \tag{2.1}$$

where $\alpha_j = \exp\frac{\pi i j}{n}$ are the roots of unity, and

$$\operatorname{Im}\mu_1 \geqslant \operatorname{Im}\mu_2 \geqslant \cdots \geqslant \operatorname{Im}\mu_{2n}.$$

Let $D_m$ be the regions that are simply connected, bounded away from branch points and contain none of the$\gamma_{jk}$ curves in their interiors, where

$$\gamma_{jk} = \{\lambda / \operatorname{Im}\mu_j = \operatorname{Im}\mu_k\}.$$

Let $D_{L(\lambda),m}$ denote the set of all functions $y(x,\lambda)$ square integrable with respect to $x \in (-\infty,\infty)$, $\lambda \in D_m$, $m = 1,\dots,2n$, and