Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 339 (2008) 469–481

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Positive solutions of a general discrete boundary value problem ☆

Guang Zhang [a,*], Zhilin Yang [b]

[a] School of Science, Tianjin University of Commerce, Tianjin 300134, PR China
[b] Department of Mathematics, Qingdao Technological University, No. 11, FuShun Road, Qingdao 266033, PR China

Received 16 June 2006

Available online 20 July 2007

Submitted by P.J. McKenna

Abstract

In this paper, the existence of positive solutions for a nonlinear general discrete boundary value problem is established. Such results extend and improve some known facts for the two-point and three-point boundary value problems. Particularly, the boundary value conditions can be nonlinear and the method is new. For explaining the main results, some numerical examples are also given.
© 2007 Elsevier Inc. All rights reserved.

Keywords: Reaction–diffusion; Steady state distribution; Positive solution; Fixed point theorem

1. Introduction

It is well know that the partial difference equation of the form

$$\Delta_1 u_k^t = r\Delta_2^2 u_{k-1}^t + f_k\big(u_k^t\big) \tag{1}$$

can describe the reaction–diffusion problem, see the monograph [1], where $\Delta_1 u_k^t = u_k^{t+1} - u_k^t$ and $\Delta_2^2 u_{k-1}^t = u_{k+1}^t - 2u_k^t + u_{k-1}^t$. Let u_k^t be the temperature of the body at the integral position k and the integral time t. At the same time, we assume that there are two ice mountains in the positions 0 and $n+1$, respectively. Naturally, we have

$$\begin{cases} \Delta_1 u_k^t = r\Delta_2^2 u_{k-1}^t + f_k\big(u_k^t\big), & k = 1, 2, \dots, n,\ t = 0, 1, \dots, \\ u_0^t = 0 = u_{n+1}^t, & t = 0, 1, \dots. \end{cases} \tag{2}$$

If the diffusion is on a ring, we can obtain the periodic initial boundary problem

$$\begin{cases} \Delta_1 u_k^t = r\Delta_2^2 u_{k-1}^t + f_k\big(u_k^t\big), & k = 1, 2, \dots, n,\ t = 0, 1, \dots, \\ u_0^t = u_n^t, \quad u_1^t = u_{n+1}^t, & t = 0, 1, \dots. \end{cases} \tag{3}$$

☆ Supported by the Natural Science Foundation of Qingdao Technological University.

* Corresponding author.
E-mail addresses: shzhg@qtech.edu.cn, qd_gzhang@126.com (G. Zhang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.07.011

The problems (2) and (3) have been extensively studied by a number of authors, see [1] and references therein. However, in the diffusion we usually hope to observe the temperature of some bodies. For example, let $m, l \in [1, n] = \{1, 2, \ldots, n\}$ and we hope to shadow u_m and u_l. Naturally, we can assume that the boundary value conditions

$$u_0^t = g(u_m^t), \qquad u_{n+1}^t = h(u_l^t).$$

Thus, we have

$$\begin{cases} \Delta_1 u_k^t = r\Delta_2^2 u_{k-1}^t + f_k(u_k^t), & k = 1, 2, \ldots, n,\ t = 0, 1, \ldots, \\ u_0^t = g(u_m^t), \quad u_{n+1}^t = h(u_l^t), & t = 0, 1, \ldots. \end{cases} \tag{4}$$

The existence and uniqueness of solutions of the system (4) is easy to see. Indeed, if the initial real distribution $\{u_k^0\}_{k=1}^n$ is known, then we may calculate successively the sequence

$$u_0^0, u_{n+1}^0; u_1^1, u_2^1, \ldots, u_n^1, u_0^1, u_{n+1}^1; \ldots$$

in a unique manner, which will give rise to a unique real solution $\{u_k^t\}_{t\in N, k\in[0,n+1]}$.

If the lateral system is insulated, the steady temperature in the system (4) will satisfy the equation

$$r\Delta^2 u_{k-1} + f_k(u_k) = 0, \quad k = 1, 2, \ldots, n, \tag{5}$$

or

$$\Delta^2 u_{k-1} + \lambda f_k(u_k) = 0, \quad k = 1, 2, \ldots, n, \tag{6}$$

with the boundary value conditions

$$u_0 = g(u_m), \qquad u_{n+1} = h(u_l). \tag{7}$$

The system (6)–(7) can be called a *general discrete boundary value problem*. Indeed, if we assume $g(u_m) = 0 = h(u_l)$, then (6)–(7) reduces to the Dirichlet problem; when $m = n$, $l = 1$, $g(u_n) = u_0$ and $h(u_1) = u_{n+1}$, (6)–(7) becomes the periodic boundary value problem; $g(u_m) = 0$ or $h(u_l) = 0$ implies that (6)–(7) is the three-point boundary problem.

It is well known that the two-point boundary value problems have been extensively studied. In fact, it is of interest to note here that the three-point or multi-point boundary value problems in the continuous case have been extensively studied in the recent papers [2–11] since the early 1980s. Recently, in [12] we have considered the existence of positive solutions for the nonlinear discrete three-point boundary value problem

$$\begin{cases} \Delta^2 x_{k-1} + f(x_k) = 0, & k \in [1, n], \\ x_0 = 0, \quad ax_l = x_{n+1}, \end{cases} \tag{8}$$

where $n \in \{2, 3, \ldots\}$, $l \in [1, n] = \{1, 2, \ldots, n\}$, and $f \in C(R^+, R^+)$. For Eq. (8), the existence of one or two positive solutions was established when f is superlinear or sublinear.

In this paper, we will consider the existence of positive solutions for (6)–(7). By a solution u of (6)–(7), we mean a real sequence u which is defined on $[0, n+1]$ and satisfies (6) with the boundary value condition (7). A solution $\{u_k\}_{k=0}^{n+1}$ of (6)–(7) is called to be positive if $u_k \geqslant 0$ for $k \in [1, n]$ and $u_1 + u_2 + \cdots + u_n > 0$.

It is well known that the Green functions are important for solving the boundary value problems. However, it is worthwhile to point out that we will not try to construct a new Green function for our boundary value problem, unlike the listed references. In Section 2, we will give some preliminary facts which are useful for instructing our main results. In Section 3, the main results will be obtained for the sublinear and superlinear cases of f_k with $k \in [1, n]$. Finally, in Section 4, some applications will be considered. Particularly, the explanatory statement is given.

2. Preliminaries

In order to obtain our main results, we present some preliminary results in this section. Let

$$g_{ij} = \begin{cases} \frac{j(n+1-i)}{n+1}, & 0 \leqslant j \leqslant i \leqslant n+1, \\ \frac{i(n+1-j)}{n+1}, & 0 \leqslant i \leqslant j \leqslant n+1. \end{cases} \tag{9}$$