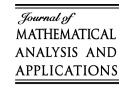


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Two-point Taylor expansions in the asymptotic approximation of double integrals. Application to the second and fourth Appell functions

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Abstract

The main difficulty in Laplace's method of asymptotic expansions of double integrals is originated by a change of variables. We consider a double integral representation of the second Appell function $F_2(a, b, b', c, c'; x, y)$ and illustrate, over this example, a variant of Laplace's method which avoids that change of variables and simplifies the computations. Essentially, the method only requires a Taylor expansion of the integrand at the critical point of the phase function. We obtain in this way an asymptotic expansion of $F_2(a, b, b', c, c'; x, y)$ for large b, b', c and c'. We also consider a double integral representation of the fourth Appell function $F_4(a, b, c, d; x, y)$. We show, in this example, that this variant of Laplace's method is uniform when two or more critical points coalesce or a critical point approaches the boundary of the integration domain. We obtain in this way an asymptotic approximation of $F_4(a, b, c, d; x, y)$ for large values of a, b, c and d. In this second example, the method requires a Taylor expansion of the integrand at two points simultaneously. For this purpose, we also investigate in this paper Taylor expansions of two-variable analytic functions with respect to two points, giving Cauchy-type formulas for the coefficients of the expansion and details about the regions of convergence.

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1. Introduction

We consider double integrals of the form

$$F(z) = \iint_{\mathbb{R}^2} e^{zh(x,y)} f(x,y) dx dy, \tag{1}$$

where D is a bounded or unbounded domain in \mathbb{R}^2 , f and h are infinitely differentiable in \bar{D} and z is a large positive parameter. Laplace's method tells us that the main contribution to the asymptotic behaviour of (1) comes from the

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points of the domain D where the phase function h(x, y) attains its largest value [9, Chapter 8]. For instance, if h(x, y) attains its maximum value only at a point $(x_0, y_0) \in D^0$, where f(x, y) and h(x, y) are infinitely differentiable, then $\nabla h(x_0, y_0) = (0, 0)$ and the Hessian matrix of h at that point, $Hh(x_0, y_0)$, is negative definite. Laplace's result is

$$F(z) \sim \frac{\pi f(x_0, y_0)}{z\sqrt{|\text{Det}[Hh(x_0, y_0)]|}} e^{zh(x_0, y_0)}, \quad z \to \infty.$$
 (2)

The right-hand side above is the first term of a complete asymptotic expansion that can be obtained in the following way [9, Chapter 8, Section 10]. The Hessian matrix $Hh(x_0, y_0)$ can be diagonalized after an orthogonal change of variables. Then, without loss of generality, we may assume that the Taylor expansion of h(x, y) at (x_0, y_0) has the form

$$h(x, y) = h(x_0, y_0) + a(x - x_0)^2 + b(y - y_0)^2 + \cdots$$

= $h(x_0, y_0) + a(x - x_0)^2 [1 + P(x, y)] + b(y - y_0)^2 [1 + Q(x, y)],$ (3)

where a, b < 0, P(x, y) and Q(x, y) are infinitely differentiable at (x_0, y_0) and $P(x_0, y_0) = Q(x_0, y_0) = 0$. Perform in (1) the change of integration variables $(x, y) \to (u, v)$ defined by $u = (x - x_0)\sqrt{1 + P(x, y)}$, $v = (y - y_0)\sqrt{1 + Q(x, y)}$,

$$F(z) = e^{zh(x_0, y_0)} \iint_{D'} e^{z(au^2 + bv^2)} g(u, v) du dv,$$
(4)

where D' is the image of D under this change of variables,

$$g(u,v) \equiv f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)}$$
(5)

and $\partial(x, y)/\partial(u, v)$ is the Jacobian of the transformation $(x, y) \to (u, v)$. If g(u, v) has a Taylor expansion at (u, v) = (0, 0),

$$g(u,v) \sim \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{m,n-m} u^{m} v^{n-m},$$
(6)

then we can apply Watson's Lemma to the integral (4): replace (6) in the right-hand side of (4) and interchange sum and integral [9, Chapter 8, Section 10],

$$F(z) \sim e^{zh(x_0, y_0)} \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{2m, 2n-2m} \frac{\Gamma(m+1/2)\Gamma(n-m+1/2)}{(-a)^{m+1/2}(-b)^{n-m+1/2}z^{n+1}}, \quad z \to \infty.$$
 (7)

Therefore, the computation of the coefficients $c_{m,n}$ in the standard Laplace method is very difficult because of the complexity of the above mentioned change of variable (see the example in [9, Chapter 8, Section 10] where the first term of the expansion of a double integral is derived). In general, only a few terms of the expansion are obtained explicitly.

In [2], a modification of Laplace's method for one-dimensional integrals is proposed which avoids the change of variable and simplifies the computation of the coefficients of the expansion. Consider the integral

$$F(x) \equiv \int_{a}^{b} e^{xh(t)} f(t) dt, \tag{8}$$

where (a, b) is a real interval (finite or infinite), x is a large positive parameter and f(t) and h(t) are infinitely differentiable in (a, b). It is shown in [2] that it is not necessary to use a change of variables in (8) to convert the integral into the Laplace form and then apply the standard Laplace method. It is just necessary to expand f(t) in (8) at the maximum of h(t) in [a, b], say t_0 , and interchange sum and integral. If f(t) has a Taylor expansion at $t = t_0$,

$$f(t) \sim \sum_{n=0}^{\infty} f_n (t - t_0)^n,$$
 (9)

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