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Two-point Taylor expansions in the asymptotic approximation of double integrals. Application to the second and fourth Appell functions

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Abstract

The main difficulty in Laplace's method of asymptotic expansions of double integrals is originated by a change of variables. We consider a double integral representation of the second Appell function $F_2(a, b, b', c, c'; x, y)$ and illustrate, over this example, a variant of Laplace's method which avoids that change of variables and simplifies the computations. Essentially, the method only requires a Taylor expansion of the integrand at the critical point of the phase function. We obtain in this way an asymptotic expansion of $F_2(a, b, b', c, c'; x, y)$ for large b, b', c and c'. We also consider a double integral representation of the fourth Appell function $F_4(a, b, c, d; x, y)$. We show, in this example, that this variant of Laplace's method is uniform when two or more critical points coalesce or a critical point approaches the boundary of the integration domain. We obtain in this way an asymptotic approximation of $F_4(a, b, c, d; x, y)$ for large values of a, b, c and d. In this second example, the method requires a Taylor expansion of the integrand at two points simultaneously. For this purpose, we also investigate in this paper Taylor expansions of two-variable analytic functions with respect to two points, giving Cauchy-type formulas for the coefficients of the expansion and details about the regions of convergence.

Keywords: The second and fourth Appell hypergeometric functions; Uniform asymptotic expansions; Two-point Taylor expansions

1. Introduction

We consider double integrals of the form

$$F(z) = \iint_D e^{zh(x,y)} f(x, y)\, dx\, dy, \tag{1}$$

where D is a bounded or unbounded domain in $\mathbb{R}^2$, f and h are infinitely differentiable in $\bar{D}$ and z is a large positive parameter. Laplace's method tells us that the main contribution to the asymptotic behaviour of (1) comes from the

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points of the domain D where the phase function $h(x, y)$ attains its largest value [9, Chapter 8]. For instance, if $h(x, y)$ attains its maximum value only at a point $(x_0, y_0) \in D^0$, where $f(x, y)$ and $h(x, y)$ are infinitely differentiable, then $\nabla h(x_0, y_0) = (0, 0)$ and the Hessian matrix of h at that point, $Hh(x_0, y_0)$, is negative definite. Laplace's result is

$$F(z) \sim \frac{\pi f(x_0, y_0)}{z\sqrt{|\mathrm{Det}[Hh(x_0, y_0)]|}} e^{zh(x_0,y_0)}, \quad z \to \infty. \tag{2}$$

The right-hand side above is the first term of a complete asymptotic expansion that can be obtained in the following way [9, Chapter 8, Section 10]. The Hessian matrix $Hh(x_0, y_0)$ can be diagonalized after an orthogonal change of variables. Then, without loss of generality, we may assume that the Taylor expansion of $h(x, y)$ at (x_0, y_0) has the form

$$\begin{aligned} h(x, y) &= h(x_0, y_0) + a(x - x_0)^2 + b(y - y_0)^2 + \cdots \\ &= h(x_0, y_0) + a(x - x_0)^2\big[1 + P(x, y)\big] + b(y - y_0)^2\big[1 + Q(x, y)\big], \end{aligned} \tag{3}$$

where $a, b < 0$, $P(x, y)$ and $Q(x, y)$ are infinitely differentiable at (x_0, y_0) and $P(x_0, y_0) = Q(x_0, y_0) = 0$. Perform in (1) the change of integration variables $(x, y) \to (u, v)$ defined by $u = (x - x_0)\sqrt{1 + P(x, y)}$, $v = (y - y_0)\sqrt{1 + Q(x, y)}$,

$$F(z) = e^{zh(x_0,y_0)} \iint_{D'} e^{z(au^2+bv^2)} g(u, v)\, du\, dv, \tag{4}$$

where D' is the image of D under this change of variables,

$$g(u, v) \equiv f\big(x(u, v), y(u, v)\big)\frac{\partial(x, y)}{\partial(u, v)} \tag{5}$$

and $\partial(x, y)/\partial(u, v)$ is the Jacobian of the transformation $(x, y) \to (u, v)$. If $g(u, v)$ has a Taylor expansion at $(u, v) = (0, 0)$,

$$g(u, v) \sim \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{m,n-m} u^m v^{n-m}, \tag{6}$$

then we can apply Watson's Lemma to the integral (4): replace (6) in the right-hand side of (4) and interchange sum and integral [9, Chapter 8, Section 10],

$$F(z) \sim e^{zh(x_0,y_0)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{2m,2n-2m} \frac{\Gamma(m + 1/2)\Gamma(n - m + 1/2)}{(-a)^{m+1/2}(-b)^{n-m+1/2} z^{n+1}}, \quad z \to \infty. \tag{7}$$

Therefore, the computation of the coefficients $c_{m,n}$ in the standard Laplace method is very difficult because of the complexity of the above mentioned change of variable (see the example in [9, Chapter 8, Section 10] where the first term of the expansion of a double integral is derived). In general, only a few terms of the expansion are obtained explicitly.

In [2], a modification of Laplace's method for one-dimensional integrals is proposed which avoids the change of variable and simplifies the computation of the coefficients of the expansion. Consider the integral

$$F(x) \equiv \int_a^b e^{xh(t)} f(t)\, dt, \tag{8}$$

where (a, b) is a real interval (finite or infinite), x is a large positive parameter and $f(t)$ and $h(t)$ are infinitely differentiable in (a, b). It is shown in [2] that it is not necessary to use a change of variables in (8) to convert the integral into the Laplace form and then apply the standard Laplace method. It is just necessary to expand $f(t)$ in (8) at the maximum of $h(t)$ in $[a, b]$, say t_0, and interchange sum and integral. If $f(t)$ has a Taylor expansion at $t = t_0$,

$$f(t) \sim \sum_{n=0}^{\infty} f_n(t - t_0)^n, \tag{9}$$