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The uniqueness theorems of meromorphic functions sharing three values and one pair of values ☆

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Abstract

In this paper, we deal with a uniqueness theorem of two meromorphic functions that have three weighted sharing values and one pair of values. The results in this paper improve those given by G.G. Gundersen, G. Brosch, T.C. Alzahary, T.C. Alzahary and H.X. Yi, I. Lahiri and P. Sahoo, and other authors.

Keywords: Meromorphic functions; Weighted sharing values; Uniqueness theorems

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$, we denote by $S(r,h)$ any quantity satisfying

$$S(r,h)=o\big(T(r,h)\big) \quad (r\to\infty,\ r\notin E).$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. Similarly, we say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (see [16]). We say that $a(z)$ is a small function of f, if $a(z)$ is a meromorphic function satisfying $T(r,a(z))=o(T(r,f))$ $(r\notin E)$, as $r\to\infty$. In addition, we need the following definition.

Definition 1.1. (See [1, Definition 1].) Let p be a positive integer and $a\in C\cup\{\infty\}$. Then by $N_{p)}(r,\frac{1}{f-a})$ we denote the counting function of those zeros of $f-a$ (counted with proper multiplicities) whose multiplicities are not

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greater than p, by $\overline{N}_{p)}(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r, \frac{1}{f-a})$ we denote the counting function of those zeros of $f-a$ (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities).

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let a be a value in the extended plane. Let S be a subset of distinct elements in the extended plane. Next we define

$$E_f(S) = \bigcup_{a \in S} \{z\colon\ f(z) = a\},$$

where each a-point of f with multiplicity m is repeated m times in $E_f(S)$ (see[5]). Similarly, we define

$$\overline{E}_f(S) = \bigcup_{a \in S} \{z\colon\ f(z) = a\},$$

where each point in $\overline{E}_f(\{a\})$ is counted only once. We say that f and g share the set S CM, provided $E_f(S) = E_g(S)$. We say that f and g share the set S IM, provided $\overline{E}_f(S) = \overline{E}_g(S)$. Let k be a positive integer, we denote by $\overline{E}_{k)}(a, f)$ the set of zeros of $f(z) - a$ with multiplicity $\leqslant k$, and each such zero of $f(z) - a$ is counted only once (see [2, Definition 3]).

In 1926, R. Nevanlinna proved the following theorem.

Theorem A. *(See [15].) If f and g are distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3 and a_4 CM, then f is a Möbius transformation of g, two of the shared values, say a_1 and a_2, are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In 1979, G.G. Gundersen proved the following theorem, which improved Theorem A.

Theorem B. *(See [6, Theorem 1].) Let f and g be two distinct nonconstant meromorphic functions such that f and g share three values CM and share a fourth value IM, then f and g share all four values CM, and hence the conclusion of Theorem* A *holds.*

In 1989, G. Brosch proved the following theorem, which improved Theorems A and B.

Theorem C. *(See [4].) Let f and g be two distinct nonconstant meromorphic functions such that f and g share* 0, 1 *and ∞ CM, and let a and b be two distinct finite complex numbers such that $a, b \notin \{0, 1\}$. If $f - a$ and $g - b$ share* 0 *IM, then f is a Möbius transformation of g.*

Regarding Theorem C, it is natural to ask the following question.

Question 1.1. (See [8].) Is it really impossible to relax in any way the nature of sharing any one of 0, 1 and ∞ in Theorem C?

In this paper, we will deal with Question 1.1. To this end we employ the idea of weighted sharing of values which measures how close a shared value is to being shared IM or to being shared CM. The notion is explained in the following definition.

Definition 1.2. (See [9, Definition 4].) Let k be a nonnegative integer or infinity. For any $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$, and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

Remark 1.1. Definition 1.2 implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity m ($\leqslant k$) if and only if it is a zero of $g - a$ with multiplicity m ($\leqslant k$), and z_0 is a zero of $f - a$ with