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Some geometric conditions which imply the fixed point property for multivalued nonexpansive mappings ☆

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Abstract

We show some geometric conditions on a Banach space X concerning the modulus of smoothness, the coefficient of weak orthogonality, the coefficient $R(a, X)$, the James constant and the Jordan–von Neumann constant, which imply the existence of fixed points for multivalued nonexpansive mappings. These fixed point theorems improve some previous results and give affirmative answers to some open questions.

Keywords: Multivalued nonexpansive mapping; Fixed point; Normal structure; Modulus of smoothness; Coefficient of weak orthogonality; James constant; Jordan–von Neumann constant

1. Introduction

The Banach Contraction Principle was extended to multivalued contractive mappings in complete metric spaces by S.B. Nadler [16] in 1969. From then on, some other classical fixed point theorems for singlevalued mappings have been extended to multivalued mappings. However many questions remain open, for instance, the possibility of extending the well-known Kirk's theorem [12], that is, do Banach spaces with weak normal structure have the fixed point property (FPP) for multivalued nonexpansive mappings?

Since weak normal structure is implied by different geometrical properties of Banach spaces, it is natural to study if those properties imply the FPP for multivalued mappings. S. Dhompongsa et al. [2,3] introduced the Domínguez–Lorenzo condition ((DL)-condition, in short) and property (D) which imply the FPP for multivalued nonexpansive mappings. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply either the (DL)-condition or property (D). In this setting the following results have been obtained:

(a) S. Dhompongsa et al. [2, Theorem 3.4] proved that uniformly nonsquare Banach spaces with property WORTH satisfy the (DL)-condition.

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(b) S. Dhompongsa et al. [3, Theorem 3.7] showed that the condition

$$C_{NJ}(X) < 1 + \frac{WCS(X)^2}{4}$$

implies property (D).

(c) T. Domínguez Benavides and B. Gavira [5, Theorem 2] proved that if

$$\xi_X(\beta) < \frac{1}{1-\beta} \quad \text{for some } \beta \in (0,1),$$

then X satisfies the (DL)-condition. Consequently, Banach spaces such that $\rho'_X(0) < 1/2$ (in particular, uniformly smooth Banach spaces) satisfy the (DL)-condition.

(d) S. Saejung [17, Theorem 3] showed that a Banach space X has property (D) whenever $\varepsilon_0(X) < WCS(X)$ (this result improves [3, Theorem 3.7]).

(e) A. Kaewkhao [10, Theorem 3.1] proved that a Banach space X with

$$J(X) < 1 + \frac{1}{\mu(X)}$$

satisfies the (DL)-condition (this result improves [2, Theorem 3.4]). He also showed that the condition

$$C_{NJ}(X) < 1 + \frac{1}{\mu(X)^2}$$

implies the (DL)-condition [10, Theorem 4.1].

Recently, E.M. Mazcuñán-Navarro [15] has established new lower bounds for the weakly convergent sequence coefficient $WCS(X)$ of a Banach space X, in terms of the modulus of smoothness, the coefficient of weak orthogonality, the coefficient $R(a, X)$, the James constant and the Jordan–von Neumann constant. By mean of these bounds, she identifies new geometrical properties which imply weak normal structure. In this paper we show that some of those properties imply the (DL)-condition and so the FPP for multivalued nonexpansive mappings. We give different examples which show that our results are strictly more general than some previous results on this subject. Furthermore, in Corollaries 1 and 2 we give affirmative answers to open questions which appear in [2].

2. Preliminaries

We are going to recall some concepts and results which will be used in the following sections. For more details the reader may consult for instance [1,8] or [18].

Let X be a Banach space and C be a nonempty subset of X. We shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T: C \to CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leqslant \|x - y\|, \quad x, y \in C,$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max\Big\{ \sup_{x\in A} \inf_{y\in B} \|x-y\|, \sup_{y\in B} \inf_{x\in A} \|x-y\| \Big\}, \quad A, B \in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r(C, \{x_n\})$ and the asymptotic center $A(C, \{x_n\})$ of $\{x_n\}$ in C are defined by

$$r\big(C, \{x_n\}\big) = \inf\Big\{ \limsup_n \|x_n - x\|: \ x \in C \Big\}$$

and

$$A\big(C, \{x_n\}\big) = \Big\{ x \in C: \ \limsup_n \|x_n - x\| = r\big(C, \{x_n\}\big) \Big\},$$

respectively. It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set whenever C is.