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On sharp triangle inequalities in Banach spaces ☆

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Abstract

We shall present a couple of norm inequalities which will much improve the sharp triangle inequality with n elements and its reverse inequality in a Banach space shown recently by the last three authors.

Keywords: Inequality; Triangle inequality

1. Introduction and results

The triangle inequality is one of the most fundamental inequalities in analysis and have been treated by many authors (e.g., [1–3,8–10], etc.). Recently Kato, Saito and Tamura [5] showed the following sharp triangle inequality and its reverse inequality with n elements in a Banach space.

Theorem A. *(See [5].) For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X,*

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$$\left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\leqslant j\leqslant n} \|x_j\|$$

$$\leqslant \sum_{j=1}^n \|x_j\| \tag{1}$$

$$\leqslant \left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\leqslant j\leqslant n} \|x_j\|. \tag{2}$$

As the case $n = 2$ we have the following.

Theorem B. *For all nonzero elements x, y in a Banach space X with $\|x\| \geqslant \|y\|$,*

$$\|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|y\|$$

$$\leqslant \|x\| + \|y\| \tag{3}$$

$$\leqslant \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\|. \tag{4}$$

The first inequality with two elements (3) was given earlier in Hudzik and Landes [4] (see Lemma 3 below); the inequalities (3) and (4) are also found in a recent paper of Maligranda [7], while the above Theorem A was presented in [5] independently to treat the uniform non-ℓ_1^n-ness of Banach spaces (cf. [5,6]).

In the present paper we shall show the following inequalities which are sharper than the above inequalities.

Theorem 1. *For all nonzero elements $x_1, \ldots, x_n$ in a Banach space X, $n \geqslant 2$,*

$$\left\|\sum_{j=1}^n x_j\right\| + \sum_{k=2}^n \left(k - \left\|\sum_{j=1}^k \frac{x_j^*}{\|x_j^*\|}\right\|\right)\left(\|x_k^*\| - \|x_{k+1}^*\|\right)$$

$$\leqslant \sum_{j=1}^n \|x_j\| \tag{5}$$

$$\leqslant \left\|\sum_{j=1}^n x_j\right\| - \sum_{k=2}^n \left(k - \left\|\sum_{j=n-(k-1)}^n \frac{x_j^*}{\|x_j^*\|}\right\|\right)\left(\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|\right), \tag{6}$$

where $x_1^*, x_2^*, \ldots, x_n^*$ are the rearrangement of $x_1, x_2, \ldots, x_n$ satisfying $\|x_1^*\| \geqslant \|x_2^*\| \geqslant \cdots \geqslant \|x_n^*\|$, and $x_0^* = x_{n+1}^* = 0$.

As the case $n = 2$ Theorem 1 includes Theorem B. To see explicitly that Theorem 1 refines Theorem A we reformulate Theorem 1 as follows.

Theorem 1a. *For all nonzero elements $x_1, \ldots, x_n$ in a Banach space X, $n \geqslant 3$,*

$$\left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j^*}{\|x_j^*\|}\right\|\right)\|x_n^*\| + \sum_{k=2}^{n-1} \left(k - \left\|\sum_{j=1}^k \frac{x_j^*}{\|x_j^*\|}\right\|\right)\left(\|x_k^*\| - \|x_{k+1}^*\|\right)$$