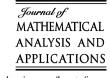




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On sharp triangle inequalities in Banach spaces *

Ken-Ichi Mitani a, Kichi-Suke Saito b,*, Mikio Kato c, Takayuki Tamura d

- a Department of Mathematics and Information Science, Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan
 - ^b Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan
 ^c Department of Mathematics, Kyushu Institute of Technology, Kitakyushu 804-8550, Japan
 - d Graduate School of Social Sciences and Humanities, Chiba University, Chiba 263-8522, Japan

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Abstract

We shall present a couple of norm inequalities which will much improve the sharp triangle inequality with n elements and its reverse inequality in a Banach space shown recently by the last three authors. © 2007 Elsevier Inc. All rights reserved.

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1. Introduction and results

The triangle inequality is one of the most fundamental inequalities in analysis and have been treated by many authors (e.g., [1-3,8-10], etc.). Recently Kato, Saito and Tamura [5] showed the following sharp triangle inequality and its reverse inequality with n elements in a Banach space.

Theorem A. (See [5].) For all nonzero elements x_1, x_2, \ldots, x_n in a Banach space X,

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Corresponding author.

E-mail addresses: mitani@m.sc.niigata-u.ac.jp (K.-I. Mitani), saito@math.sc.niigata-u.ac.jp (K.-S. Saito), katom@tobata.isc.kyutech.ac.jp (M. Kato), tamura@le.chiba-u.ac.jp (T. Tamura).

$$\left\| \sum_{j=1}^{n} x_{j} \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_{j}}{\|x_{j}\|} \right\| \right) \min_{1 \leq j \leq n} \|x_{j}\|$$

$$\leq \sum_{j=1}^{n} \|x_{j}\|$$
(1)

$$\leq \left\| \sum_{j=1}^{n} x_{j} \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_{j}}{\|x_{j}\|} \right\| \right) \max_{1 \leq j \leq n} \|x_{j}\|. \tag{2}$$

As the case n = 2 we have the following.

Theorem B. For all nonzero elements x, y in a Banach space X with $||x|| \ge ||y||$,

$$||x + y|| + \left(2 - \left\| \frac{x}{||x||} + \frac{y}{||y||} \right\| \right) ||y||$$

$$\leq ||x|| + ||y||$$
(3)

$$\leq \|x + y\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \|x\|.$$
 (4)

The first inequality with two elements (3) was given earlier in Hudzik and Landes [4] (see Lemma 3 below); the inequalities (3) and (4) are also found in a recent paper of Maligranda [7], while the above Theorem A was presented in [5] independently to treat the uniform non- ℓ_1^n -ness of Banach spaces (cf. [5,6]).

In the present paper we shall show the following inequalities which are sharper than the above inequalities.

Theorem 1. For all nonzero elements x_1, \ldots, x_n in a Banach space $X, n \ge 2$,

$$\left\| \sum_{j=1}^{n} x_{j} \right\| + \sum_{k=2}^{n} \left(k - \left\| \sum_{j=1}^{k} \frac{x_{j}^{*}}{\|x_{j}^{*}\|} \right\| \right) (\left\| x_{k}^{*} \right\| - \left\| x_{k+1}^{*} \right\|)$$

$$\leq \sum_{j=1}^{n} \|x_{j}\|$$
(5)

$$\leq \left\| \sum_{j=1}^{n} x_{j} \right\| - \sum_{k=2}^{n} \left(k - \left\| \sum_{j=n-(k-1)}^{n} \frac{x_{j}^{*}}{\|x_{j}^{*}\|} \right\| \right) (\|x_{n-k}^{*}\| - \|x_{n-(k-1)}^{*}\|), \tag{6}$$

where $x_1^*, x_2^*, \dots, x_n^*$ are the rearrangement of x_1, x_2, \dots, x_n satisfying $||x_1^*|| \ge ||x_2^*|| \ge \dots \ge ||x_n^*||$, and $x_0^* = x_{n+1}^* = 0$.

As the case n = 2 Theorem 1 includes Theorem B. To see explicitly that Theorem 1 refines Theorem A we reformulate Theorem 1 as follows.

Theorem 1a. For all nonzero elements x_1, \ldots, x_n in a Banach space $X, n \ge 3$,

$$\left\| \sum_{j=1}^{n} x_{j} \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_{j}^{*}}{\|x_{j}^{*}\|} \right\| \right) \|x_{n}^{*}\| + \sum_{k=2}^{n-1} \left(k - \left\| \sum_{j=1}^{k} \frac{x_{j}^{*}}{\|x_{j}^{*}\|} \right\| \right) (\|x_{k}^{*}\| - \|x_{k+1}^{*}\|)$$

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