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On a Dirichlet problem with $p(x)$-Laplacian

Marek Galewski

Faculty of Mathematics and Computer Science, University of Lodz, Banacha 22, 90-238 Lodz, Poland

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Abstract

We show the existence and stability of solutions for a family of Dirichlet problems

$$-\operatorname{div}\big(V_{z_1}(x,\nabla u),\dots,V_{z_N}(x,\nabla u)\big)+L_u(x,u)=F_u^k(x,u),$$
$$u\in W_0^{1,p(x)}(\Omega)$$

in a bounded domain and with nonconvex nonlinearity satisfying some local growth conditions. The conditions upon V and L allow for considering the $p(x)$-Laplacian equation. We use the relations between critical points and critical values to the primal and a suitable dual action functional to get the existence, stability and some properties of the solutions.

Keywords: $p(x)$-Laplacian; Duality; Existence; Variational method; Stability of solutions

1. Introduction

In [3] the existence of weak solutions for the following Dirichlet problem

$$-\operatorname{div}\big(a(x)\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\big)+b(x)\big|u(x)\big|^{p(x)-2}u(x)=f\big(x,u(x)\big),$$
$$u(x)|_{\partial\Omega}=0,\quad u\in W_0^{1,p(x)}(\Omega) \tag{1}$$

is considered. Here $\Omega\subset R^N$ is a bounded region, $p,q\in C(\overline{\Omega})$, $1/p(x)+1/q(x)=1$ for $x\in\Omega$; $W_0^{1,p(x)}(\Omega)$ denotes the generalized Orlicz–Sobolev space, see [5,7]; $a,b\in L^\infty(\Omega)$

E-mail address: galewski@math.uni.lodz.pl.

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with $a(x) \geqslant a_0 > 0$ and $b(x) \geqslant b_0 \geqslant 0$ a.e. on Ω for $k = 0, 1, 2, \ldots$. Let $p^- = \inf_{x\in\Omega} p(x) > 1$, $p^+ = \sup_{x\in\Omega} p(x) < N$. Let

$$p^*(x) = \begin{cases} Np(x)/\big(N - p(x)\big), & p(x) < N, \\ +\infty, & p(x) \geqslant N. \end{cases}$$

The nonlinearity satisfies either $f(x,u) = g(x)u^{\alpha(x)}$ where $p(x) - 1 < \alpha(x) < p^*(x) - 1$ or $f(x,u) = h(x)u^{\beta(x)}$ where $0 \leqslant \beta(x) < p(x) - 1$. The authors apply the mountain pass theorem to get the existence of positive weak solutions considering the $p(x)$-Laplacian operator in a more general setting than in [6] where also by the mountain pass theorem the existence of solutions for the Dirichlet problem

$$-\operatorname{div}\big(\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\big) = F_x\big(x, u(x)\big),$$
$$u(x)|_{\partial\Omega} = 0, \quad u \in W_0^{1,p(x)}(\Omega)$$

is proved in a superlinear cases. In both sources the direct method of the calculus of variations is applied in sublinear case. The nonlinearity in [6] is always assumed to satisfy: $F_x : \Omega \times R \to R$ is a Caratheodory function and $|f(x,u)| \leqslant C_1 + C_2|u|^{\alpha(x)-1}$, where C_1, C_2 are constants and $\alpha(x) < p^*(x)$. In a sublinear case it is assumed that $|F_x(x,u)| \leqslant C_1 + C_2|u|^{\beta-1}$, where $1 \leqslant \beta < p^{-1}$. In the superlinear case the authors assume $\exists M > 0$, $\theta > p^+$ such that $0 < \theta F(x,u) \leqslant uF(x,u)$ for $|u| \geqslant M$, $x \in \Omega$ and $F_x(x,u) = o(|u|^{p^+-1}) \to 0$ uniformly, where $\alpha^- > p^+$. Later it is shown that F satisfies a Palais–Smale condition and a mountain pass geometry is applied to get the existence of a nontrivial solution.

Such problems as studied in [3,6] are applied in elastic mechanics and electrorheological fluid dynamics, see [13,15] and references therein.

We propose to study a family of general problems for $k = 0, 1, 2, \ldots$

$$-\operatorname{div}\big(V_{z_1}\big(x, \nabla u(x)\big), \ldots, V_{z_N}\big(x, \nabla u(x)\big)\big) + L_u\big(x, u(x)\big) = F_u^k\big(x, u(x)\big),$$
$$u(x)|_{\partial\Omega} = 0 \tag{2}$$

in the space $W_0^{1,p(x)}(\Omega) \cap L^{\sigma(x)}(\Omega)$, $\Omega \subset R^N$ has a Lipschitz-continuous boundary and with suitable assumptions on V and L which are valid for the $p(x)$-Laplacian operator; V_{z_i} for $i = 1, 2, \ldots, N$ denotes the partial derivative with respect to z_i of a function $V : \Omega \times R^N \to R$.

Here $\sigma, \tau \in C(\overline{\Omega})$, $1/\sigma(x) + 1/\tau(x) = 1$ for $x \in \Omega$ with $\sigma^- > 1$; $p, q \in C(\overline{\Omega})$, $1/p(x) + 1/q(x) = 1$ for $x \in \Omega$. We assume that $p^- > N > 2$. Following some ideas from [9] and [10] we construct a dual variational method which will provide the existence of solutions together with some of their properties. Therefore we think that our approach being quite different and thus allowing for non restricted nonlinearities may be of some interest. The method from [9] may not be applied directly due to the more general differential operator than the one used in [6]. Indeed, the operator in Eq. (2) is not in the "divergence form" and if we try to rewrite Eq. (2) in order to get the "divergence form," we obtain the convex-concave nonlinearity. Both such cases may not be treated by the dual variational method directly. Also we use less restrictive growth assumptions and propose a different approach towards the existence of solutions. Therefore as in [10] two dual variables are introduced and consequently a different dual action functional must be investigated. Relating critical values and critical points to the action functional and the dual action functional on a specially constructed subsets of their domains allows not only for obtaining solutions but also provides some their properties.