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A remark on precomposition on $\mathrm{H}^{1/2}(S^1)$ and ε-identifiability of disks in tomography

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Abstract

We consider the inverse conductivity problem with one measurement for the equation

$$\operatorname{div}\big(\big(\sigma_1 + (\sigma_2 - \sigma_1)\chi_\omega\big)\nabla u\big) = 0$$

determining the unknown inclusion ω included in Ω. We suppose that Ω is the unit disk of $\mathbb{R}^2$. With the tools of the conformal mappings, of elementary Fourier analysis and by studying how $\mathrm{W}^{1,\infty}(S^1, S^1)$ diffeomorphisms act by precomposition on the Sobolev space $\mathrm{H}^{1/2}(S^1)$, we show how to approximate the Dirichlet-to-Neumann map when the original inclusion ω is a ε-approximation of a disk. This enables us to give some uniqueness and stability results.

Keywords: Inverse problem of conductivity; Dirichlet-to-Neumann map; Conformal mapping; Fourier series; Precomposition in Sobolev spaces

1. Introduction

In this paper, we study the inverse problem of conductivity with one measurement. Given a bounded domain $\Omega \subset \mathbb{R}^2$ with reasonably smooth boundary, a connected open set ω strictly contained in Ω, we consider for any $f \in \mathrm{H}^{1/2}(\partial\omega)$ the problem of recovering the subset ω entering the Dirichlet equation

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$$P[\omega, f] \quad \begin{cases} \operatorname{div}((\sigma_1 + (\sigma_2 - \sigma_1)\chi_\omega)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1}$$

from the knowledge of the current flux $g = \sigma_1 \partial_n u$ in $\partial\Omega$ induced by the boundary value $f = u|_{\partial\Omega}$. We will denote $\Lambda_\omega : \mathrm{H}^{1/2}(\partial\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ the Dirichlet-to-Neumann map which maps the Dirichlet data f onto the corresponding Neumann data $g = \Lambda_\omega(f) = \sigma_1 \partial_n u$. We can reformulate the inverse problem with one measurement as the determination of ω from the Cauchy pair (f, g). We mention here that we do not need the full knowledge of the Dirichlet-to-Neumann map but only one pair of Cauchy data (f, g). For such a problem, we know that the uniqueness question is, in general, an open problem. It has been solved only for the special class of convex polyhedra [12], disks and balls [13]. For other domains, Fabes, Kang and Seo [6] have studied the global uniqueness and stability within the class of domains which are ε-perturbations of disks. The main ingredients in the paper [6] were layer potential techniques and a representation formula for the solution u_ω of the problem $P[\omega, f]$.

Our main goal is to revisit the paper [6] with other techniques than boundary integral representations. Throughout our paper, the two-dimensional case will be considered. Instead of using layer potential techniques, conformal mappings and Fourier analysis will be another approach to review the two questions of stability and uniqueness within the class of disks and perturbed disks. We have to emphasize that our new approach does not allow to fully recover the results obtained by Fabes et al. in [6]. Nevertheless, during the proof of approximated identifiability, we will prove results that have their own interest.

Let us illustrate briefly the main steps of our arguments. Since $\Omega \setminus \bar{\omega}$ is doubly connected, conformal mappings allow the construction of the conformal transplant function which is solution of an elliptic problem that is obtained by transporting the original problem $P[\omega, f]$ by means of a change of variables induced by the conformal mappings. A natural way to study the original Dirichlet-to-Neumann map is to study the transplanted Dirichlet-to-Neumann map. Indeed, when the original inclusion ω is a disk in Ω, we can give an expression for the new Dirichlet-to-Neumann map by means of Moebius transforms and the explicit formula of the Dirichlet-to-Neumann operator related to a concentric annulus. The elementary properties of Moebius transforms allow us to get an uniqueness result within the class of circular inclusions and for some special Dirichlet boundary measurement.

When ω is not a disk, things become more difficult. The conformal transplantation furnishes a Dirichlet-to-Neumann operator that is not very convenient to study. Indeed, there is no way to obtain an explicit expression for this operator. However, when the original inclusion ω is an ε-perturbation of a disk D, then we have a reliable expression $\Lambda_\omega = \Lambda_D + R_\varepsilon$ with a remainder R_ε that is of order ε^δ where $\delta \in (0, \gamma)$ is arbitrary when the regularity of the unknown boundary $\partial\omega$ is $\mathcal{C}^{2,\gamma}$, $0 < \gamma < 1$. In the conformal transplant, we have to deal with the two conformal mappings that map respectively $\Omega \setminus \bar{\omega}$ into the annulus and ω on its inner ball. The restriction of the maps on the corresponding boundaries will be of great importance to control R_ε.

The estimate of $\|R_\varepsilon\|_{\mathcal{L}(\mathrm{H}^{1/2}(\partial\Omega), \mathrm{H}^{-1/2}(\partial\Omega))}$ is not straightforward: we tackle it by estimating $\|h - h \circ \xi\|_{\mathrm{H}^{1/2}(S^1)}$ when $\xi : S^1 \mapsto S^1$ is a $\mathrm{W}^{1,\infty}$ diffeomorphism of the circle and when h is a function that belongs to some Sobolev space $\mathrm{H}^s(S^1)$, $s > \frac{1}{2}$. In our context, such a diffeomorphism ξ is obtained from a composition of two boundary correspondence functions. We are not able to give the best Sobolev exponent s for which the estimate is true. However we give a result for the exponent values $s = 1 + \alpha$ for any $0 < \alpha < 1$. At our best knowledge, the question remains open when h belongs to $\mathrm{H}^s(S^1)$ when $\frac{1}{2} < s < 1$. Our result about the precomposition of Sobolev spaces is essentially inspired from the papers [3,4,16] where are studied the action of quasi-symmetric homeomorphisms on the critical Sobolev space $\mathrm{H}^{1/2}(S^1)$.