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The essential norm of a composition operator mapping into the Q_s-space ☆

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Abstract

An asymptotic formula, in terms of a global integral condition, for the essential norm $\|C_\varphi\|_e$ of the composition operator $C_\varphi(f) = f \circ \varphi$ mapping from the weighted Bergman space A^p_α, $1 < p \leqslant 2$, or the weighted Dirichlet space $\mathcal{D}_\alpha$ into the Möbius invariant Q_s-space is established. Moreover, it is shown that if C_φ is bounded from the classical Dirichlet space to *BMOA*, then

$$\|C_\varphi\|_e^2 = \limsup_{|a|\to 1^-} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(z)\right|^2 N_{\varphi\circ\varphi_b}(z)\, dA(z) = \limsup_{|z|\to 1^-} \sup_{b\in\mathbb{D}} N_{\varphi\circ\varphi_b}(z),$$

where $N_{\varphi\circ\varphi_b}(z)$ is the Nevanlinna counting function for $\varphi\circ\varphi_b$, and $\varphi_b(z) = (b-z)/(1-\bar{b}z)$.

Keywords: Q_s-space; Bergman space; Dirichlet space; Composition operator; Essential norm; Nevanlinna counting function

1. Introduction and results

Every analytic self-map φ of the open unit disc $\mathbb{D}$ in the complex plane induces the composition operator C_φ, defined by $C_\varphi(f) := f\circ\varphi$, acting on the space of all analytic functions in $\mathbb{D}$. It is well known that any such operator is a bounded linear operator on the classical Bergman and

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Hardy spaces. Probably the first papers strongly related to composition operators were [19,21]. For the theory of composition operators in spaces of analytic functions, see [5,23].

J.H. Shapiro [22] showed that the essential norm of C_φ, the distance of C_φ in the operator norm from compact operators, acting on the Hardy space H^2 equals to

$$\limsup_{|z|\to 1^-} \frac{N_\varphi(z)}{\log\frac{1}{|z|}}.$$

Writing $A^2_{-1} := H^2$, the space H^2 is identified with the limit space of the weighted Bergman space A^2_α as $\alpha \to -1^+$ [30]. With this in mind, P. Poggi-Corradini [20] generalized Shapiro's result for A^2_α when $\alpha \in \{-1, 0, 1\}$ by using contractive zero-divisors constructed by H. Hedenmalm [11,12]: the essential norm of C_φ acting on A^2_α equals to

$$\limsup_{|z|\to 1^-} \frac{N_{\varphi,2+\alpha}(z)}{\left(\log\frac{1}{|z|}\right)^{2+\alpha}}.$$

Here $N_{\varphi,\gamma}(z)$ denotes the generalized Nevanlinna counting function for the self-map φ, and $N_\varphi(z) := N_{\varphi,1}(z)$ is the Nevanlinna counting.

The purpose of this note is to study the essential norm of a composition operator mapping into the Möbius invariant Q_s-space. The approach taken here comes from Shapiro's work [22] on the essential norm of a composition operator acting on the Hardy space H^2. The main result is Theorem 1, which gives an asymptotic formula for the essential norm of the bounded operator C_φ acting from the weighted Bergman or Dirichlet space into Q_s-space in terms of a global integral condition. This yields a known characterization of compact composition operators. In the special case when the target space is $BMOA = Q_1$, the space of analytic functions of bounded mean oscillation, another asymptotic formula for the essential norm is given in Theorem 2. This one is similar to the formulas established by Shapiro and Poggi-Corradini. The proof of Theorem 2 yields Proposition 3 which gives characterizations of bounded and compact composition operators acting from the weighted Bergman or Dirichlet space into *BMOA*. Further, if C_φ acts from the classical Dirichlet space into *BMOA*, then the obtained formulas for the essential norm are not only asymptotic but they are exact by Corollary 4. It is also shown, as an example, that if the symbol φ is an inner function, then C_φ acting from the classical Besov space B^p, $1 < p \leqslant 2$, into *BMOA* is bounded, and the essential norm satisfies $\|C_\varphi\|_e \geqslant 2^{-1/2}$, so that C_φ is not compact.

Throughout this note functions denoted by f are always assumed to be analytic in $\mathbb{D}$, and φ denotes an analytic self-map of $\mathbb{D}$ so that $\varphi(\mathbb{D}) \subset \mathbb{D}$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α consists of those f for which

$$\|f\|_{A^p_\alpha} := \left((\alpha+1)\int_{\mathbb{D}} \left|f(z)\right|^p \left(1-|z|^2\right)^\alpha \, dA(z)\right)^{\frac{1}{p}} < \infty,$$

where dA is the element of the Lebesgue area measure normalized such that $A(\mathbb{D}) = 1$. For the theory of Bergman spaces, see [7,12]. A function f belongs to the Dirichlet type space $\mathcal{D}^p_\alpha$ if $f' \in A^p_\alpha$, and the true norm in $\mathcal{D}^p_\alpha$ is defined by $\|f\|_{\mathcal{D}^p_\alpha} := \|f'\|_{A^p_\alpha} + |f(0)|$ when $p \geqslant 1$. The classical Dirichlet space $\mathcal{D}$ is $\mathcal{D}^2_0$, the weighted Dirichlet space $\mathcal{D}_\alpha$ is $\mathcal{D}^2_\alpha$, and the classical Besov space B^p is $\mathcal{D}^p_{p-2}$. Moreover, it is well known that $A^p_\alpha = \mathcal{D}^p_{p+\alpha}$ and

$$C^{-1}\|f\|_{A^p_\alpha} \leqslant \|f\|_{\mathcal{D}^p_{p+\alpha}} \leqslant C\|f\|_{A^p_\alpha} \tag{1.1}$$

for a positive constant C depending only on p and α. See, for example, [31].