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ScienceDirect
J. Math. Anal. Appl. 333 (2007) 1024–1035
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Polar decomposition in e-rings

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Received 14 November 2006
Available online 16 January 2007
Submitted by J.A. Ball

Abstract

An e-ring is a generalization of the ring of bounded linear operators on a Hilbert space together with the subset consisting of all effect operators on that space. Associated with an e-ring is a partially ordered abelian group, called its directed group, that generalizes the additive group of bounded Hermitian operators on the Hilbert space. We prove that every element of the directed group of an e-ring has a polar decomposition if and only if every element has a carrier projection and is split by a projection into a positive and a negative part.

Keywords: e-Ring; Effect algebra; Directed group; Projection; Orthomodular poset; Fuzzy set; Polar decomposition; Comparability property; Positive and negative parts; Absolute value; Carrier projection; Signum

1. Introduction

An e-ring (R, E) is a generalization of the pair $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$ consisting of the ring $\mathbb{B}(\mathfrak{H})$ of bounded linear operators on a Hilbert space $\mathfrak{H}$ and the system $\mathbb{E}(\mathfrak{H})$ of so-called *effect operators* on $\mathfrak{H}$, i.e., positive semi-definite Hermitian operators on $\mathfrak{H}$ that are dominated by the identity operator. The *effect-ordered rings* studied in [6] are mathematically equivalent to e-rings. In [11,12],

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[1] Work of the author was partially supported by Science and Technology Assistance Agency under the contract No. APVT-51-032002, grant VEGA 2/6088/26 and Center of Excellence SAS—Physics of Information I/2/2005.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.12.013

the definition in [6] was reformulated as follows to emphasize the significance of the *effect algebra* E.

Definition 1.1. An *e-ring* is a pair (R, E) consisting of an associative ring R with unity 1 and a subset $E \subseteq R$ of elements called *effects* such that $0, 1 \in E$; $e \in E \Rightarrow 1 - e \in E$; and the set E^+ consisting of all finite sums $e_1 + e_2 + \cdots + e_n$ with $e_1, e_2, \ldots, e_n \in E$ satisfies the following conditions.

For all $a, b \in E^+$:

$$\text{(i) } -a \in E^+ \Rightarrow a = 0, \qquad \text{(ii) } 1 - a \in E^+ \Rightarrow a \in E,$$
$$\text{(iii) } ab = ba \Rightarrow ab \in E^+, \qquad \text{(iv) } aba \in E^+,$$
$$\text{(v) } aba = 0 \Rightarrow ab = ba = 0, \quad \text{and} \quad \text{(vi) } (a - b)^2 \in E^+.$$

If (R, E) is an e-ring, then the subgroup $G := \{a - b \mid a, b \in E^+\} = E^+ - E^+$ of the additive group of the ring R is called the *directed group* of (R, E), and $P := \{p \in G \mid p = p^2\}$ is called the set of *projections* in G.

For the Hilbert-space e-ring $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$, the set $\mathbb{E}(\mathfrak{H})^+$ consists of all positive semi-definite bounded Hermitian operators on $\mathfrak{H}$, the directed group $\mathbb{G}(\mathfrak{H})$ is the additive abelian group of all bounded Hermitian operators on $\mathfrak{H}$, and the set $\mathbb{P}(\mathfrak{H})$ of projections is the set of all (orthogonal) projection operators on $\mathfrak{H}$.

In our study of e-rings (R, E), we are mainly interested in the mathematical structure of the system P of projections, the "effect algebra" E, and the directed group G—the enveloping ring R is merely a convenient environment in which to conduct the study of P, E, and G. In [11], it is shown that the additive group G can be organized into a directed partially ordered abelian group [14, pp. 1–4] by defining, for $g, h \in G$,

$$g \leqslant h \quad \Leftrightarrow \quad h - g \in E^+.$$

It is also shown that $E = \{e \in G \mid 0 \leqslant e \leqslant 1\}$, so E is an *interval effect algebra* [1], and that

$$0, 1 \in P \subseteq E \subseteq E^+ \subseteq G \subseteq R.$$

Furthermore, with the partial order inherited from G, and with $p \mapsto 1 - p$ as the orthocomplementation, P is an *orthomodular poset* (OMP) [3,4].

Apart from its intrinsic mathematical interest, our study of e-rings has at least seven (overlapping) primary sources of motivation: (1) operator algebras, (2) the quantum theory of measurement, (3) algebraic logic, (4) fuzzy set theory, (5) Jordan algebras, (6) CB-groups, and (7) ordered division rings. We comment briefly on each of these.

Operator algebras. A unital C*-algebra R gives rise to an e-ring (R, E), where the directed group G is the additive group of self-adjoint elements in R, partially ordered as usual [17, Section 4.2], $E = \{e \in G \mid 0 \leqslant e \leqslant 1\}$ is the "unit interval" in G, and $E^+ = \{g^2 \mid g \in G\}$. The cases in which R is a von Neumann algebra, an AW*-algebra [19], or a Rickart C*-algebra [16], as well as the case in which R is commutative, are of special interest.

The quantum theory of measurement. In the contemporary quantum theory of measurement [2], the (possibly "fuzzy") observables are represented by (normalized) positive-operator-valued (POV) measures, i.e., measures defined on a Borel space and taking on values in a Hilbert-space