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Necessary and sufficient conditions for the existence of symmetric positive solutions of singular boundary value problems

John R. Graef [*,1], Lingju Kong

Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA

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Abstract

Necessary and sufficient conditions are obtained for the existence of symmetric positive solutions to the boundary value problem

$$\begin{aligned}&\left(|u''|^{p-1}u''\right)'' = f(t, u, u', u''), \quad t \in (0, 1),\\ &u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1.\end{aligned}$$

Applications of our results to the special case where f is a power function of u and its derivatives are also discussed. Moreover, similar conclusions for a more general higher order boundary value problem are established. Our results extend some recent work in the literature on boundary value problems for ordinary differential equations. The analysis of this paper mainly relies on fixed point theorems on cones.

Keywords: Boundary value problems; Existence; Symmetric positive solutions; Cone; Fixed point theorem; Necessary and sufficient conditions

* Corresponding author.
E-mail addresses: john-graef@utc.edu (J.R. Graef), lingju-kong@utc.edu (L. Kong).
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1. Introduction

We study the boundary value problem (BVP) consisting of the equation

$$\left(|u''|^{p-1}u''\right)'' = f(t,u,u',u''), \quad t \in (0,1), \tag{1.1}$$

and the boundary condition (BC)

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0,1, \tag{1.2}$$

where $p > 0$, $f : (0,1) \times \mathbb{D} \to \mathbb{R}^+$ is continuous with $\mathbb{D} = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^-$, $\mathbb{R}^+ = [0,\infty)$, and $\mathbb{R}^- = (-\infty, 0]$. If a function $u : [0,1] \to \mathbb{R}$ is continuous and satisfies that $u(t) = u(1-t)$ for $t \in [0,1]$, then we say that $u(t)$ is *symmetric* on $[0,1]$. By a $C^2[0,1]$ *symmetric positive solution* of BVP (1.1), (1.2), we mean a symmetric function $u \in C^2[0,1]$ such that $|u''|^{p-1}u'' \in C^2(0,1)$, $(-1)^i u^{(2i)}(t) > 0$ for $t \in (0,1)$ and $i = 0,1$, and $u(t)$ satisfies Eq. (1.1) and BC (1.2). If a $C^2[0,1]$ symmetric positive solution $u(t)$ further satisfies $|u''|^{p-1}u'' \in C^1[0,1]$, then $u(t)$ is said to be a *pseudo-*$C^3[0,1]$ *symmetric positive solution*. We will see that, if $p = 1$, then a pseudo-$C^3[0,1]$ symmetric positive solution $u(t)$ satisfies $u \in C^3[0,1]$. In this case, we say that $u(t)$ is a $C^3[0,1]$ *symmetric positive solution*. We observe that when f is independent of u'', Eq. (1.1) becomes

$$\left(|u''|^{p-1}u''\right)'' = f(t,u,u'), \quad t \in (0,1). \tag{1.3}$$

Interest in obtaining sufficient conditions for the existence of symmetric positive solutions of BVPs has been ongoing for several years. For a small sample of such work, we refer the reader to the papers of Avery and Henderson [1], Davis et al. [2], Davis et al. [3], Graef and Kong [4], Graef et al. [5], Graef et al. [6], and Henderson and Thompson [8]. Efforts to obtain necessary and sufficient conditions for the existence of positive solutions of BVPs can also be found in the literature, for example, in [4,10–16]. In particular, the BVP consisting of the equation

$$u^{(2m)} = f\left(t,u,u'',\dots,u^{(2m-2)}\right), \quad t \in (0,1),$$

and the BC

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0,\dots,m-1, \tag{1.4}$$

was considered in [12], and a special form of it with $m = 2$ and $f = f(t,u(t))$ was investigated in [15]. In [12,15], by means of Krasnosel'skii's fixed point theorem, necessary and sufficient conditions are found for the existence of positive solutions for the case when f is superlinear. However, when a p-Laplacian operator (or even more generally, a nonlinear term) is involved in the equation and/or f depends on both even and odd order derivatives of $u(t)$, there appears to be very little known about necessary and sufficient conditions for the existence of positive solutions of BVPs. To the best of our knowledge, the only results on this so far are given in [4] where the present authors studied the BVP consisting of the equation

$$\left(\phi(u'')\right)'' = f(t,u,u',u''), \quad t \in (0,1),$$

and BC (1.2) with $\phi : \mathbb{R} \to \mathbb{R}$ being an increasing homeomorphism. By developing the lower and upper solution method for the above problem, we obtained necessary and sufficient conditions for the existence of symmetric positive solutions for the case when f is sublinear.

In this paper, we apply Krasnosel'skii's fixed point theorem to obtain new necessary and sufficient conditions for the existence of $C^2[0,1]$ symmetric positive solutions to BVP (1.1), (1.2) and pseudo-$C^3[0,1]$ symmetric positive solutions to (1.3), (1.2), respectively. We also formulate the analogous results for the BVP consisting of the equation

$$\left(\left|u^{(2m-2)}\right|^{p-1}u^{(2m-2)}\right)'' = f\left(t,u,u',u'',\dots,u^{(2m-2)}\right), \quad t \in (0,1), \tag{1.5}$$